# THE STRUCTURE OF THE HYPOABELIAN GROUPS.

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1. THIS paper gives a marked simplification both in the general conceptions and in the detailed developments of the theory of the two hypoabelian groups of Jordan and of the writer's generalization * to the Galois field of order $2^n$ of the first hypoabelian group. It is important, especially for the generalization, to give these groups an abstract definition independent of the theory of "exposants d'échange," by means of which Jordan derived them. The crucial point in the simplified treatment lies in the discovery of the explicit relations

$$\sum_{i,j}^{1\ldots m} a_j^{(i)}\delta_j^{(i)} = m, \qquad \sum_{i,j}^{1\ldots m} a_j^{(i)}\delta_j^{(i)} + \alpha_1' + \beta_1' + \gamma_1' + \delta_1' = m,$$

satisfied by the substitutions of the simple sub-groups $J$ and $J_1$, respectively, but ruling out the remaining substitutions of the total hypoabelian groups $G$ and $G_1$. We may therefore avoid the dependence made † in §§ 274 and 289 upon the last book of the Traité (see §672, page 506).

Basing the investigation upon the groups $J$ and $J_1$, which are to be proved simple, and not upon $G$ and $G_1$ as in the earlier treatments, we wholly avoid the delicate analysis and calculations necessary in §§275 and 290. For the first hypoabelian group, the sub-division into cases is diminished one-half. For the second hypoabelian group, decided simplifications may be made in §§284, 286–8. Some errors have been detected; thus the groups $G$ and $G_1$ do not have the same order, as stated in Jordan, §279. §291 is wholly wrong.

2. The groups $G$ and $G_1$ are sub-groups of the *simple* ‡

* "The first hypoabelian group generalized," *The Quarterly Journal of Mathematics*, 1898.

† The indefinite references in Jordan remained an enigma to me until quite recently. Jordan himself could not recall them upon my personal request last year.

‡ Dickson: "A triply infinite system of simple groups," *The Quarterly Journal*, July, 1897.

Abelian group $H$ composed of the linear substitutions on $2m$ indices in the $GF[2^n]$,

$$(1)\qquad \begin{cases} \xi_i' = \sum_{j=1}^{m} (\alpha_j^{(i)} \xi_j + \gamma_j^{(i)} \eta_j), \\ \eta_i' = \sum_{j=1}^{m} (\beta_j^{(i)} \xi_j + \delta_j^{(i)} \eta_j), \end{cases} \qquad (i = 1, \cdots, m)$$

whose coefficients satisfy the relations:

$$(2)\qquad \begin{cases} \sum_{i=1}^{m} \begin{vmatrix} \alpha_j^{(i)} \gamma_j^{(i)} \\ \beta_j^{(i)} \delta_j^{(i)} \end{vmatrix} = 1, \quad \sum_{i=1}^{m} \begin{vmatrix} \alpha_j^{(i)} \gamma_k^{(i)} \\ \beta_j^{(i)} \delta_k^{(i)} \end{vmatrix} = 0, \\ \sum_{i=1}^{m} \begin{vmatrix} \alpha_j^{(i)} \alpha_k^{(i)} \\ \beta_j^{(i)} \beta_k^{(i)} \end{vmatrix} = 0, \quad \sum_{i=1}^{m} \begin{vmatrix} \gamma_j^{(i)} \gamma_k^{(i)} \\ \delta_j^{(i)} \delta_k^{(i)} \end{vmatrix} = 0, \end{cases}$$

$$(j, k = 1, \cdots, m; j \neq k).$$

In virtue of these relations the reciprocal* to (1) is:

$$(1)^{-1}\qquad \begin{cases} \xi_i' = \sum_{j=1}^{m} (\delta_i^{(j)} \xi_j + \gamma_i^{(j)} \eta_j) \\ \eta_i' = \sum_{j=1}^{m} (\beta_i^{(j)} \xi_j + \alpha_i^{(j)} \eta_j) \end{cases} \qquad (i = 1, \cdots, m)$$

so that we reach a set of relations $(2_1)$ by replacing in (2) $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $\gamma_j^{(i)}$, $\delta_j^{(i)}$ by respectively $\delta_i^{(j)}$, $\beta_i^{(j)}$, $\gamma_i^{(j)}$, $\alpha_i^{(j)}$.

Among the substitutions (1) occur the following (where only the indices altered are written):

$$N_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda\eta_j, \ \xi_j' = \xi_j + \lambda\eta_i;$$

$$R_{i,j,\lambda}: \quad \eta_i' = \eta_i - \lambda\xi_j, \ \eta_j' = \eta_j - \lambda\xi_i;$$

$$Q_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda\xi_j, \ \eta_j' = \eta_j - \lambda\eta_i;$$

$$T_{i,\lambda}: \quad \xi_i' = \lambda\xi_i \qquad \eta_i' = \lambda^{-1}\eta_i;$$

$$P_{i,j} = (\xi_i\xi_j)(\eta_i\eta_j); \ M_iM_j = (\xi_i\eta_i)(\xi_j\eta_j).$$

## PART I.—THE GROUP $J$, §§ 3–10.

3. Consider the group generated as follows:

$$J = \{M_iM_j, \ N_{i,j,\lambda} \quad (i, j = 1, \cdots, m; i \neq j)\},$$

where $\lambda$ runs through all the quantities of the $GF[2^n]$. $J$ contains $Q_{i,j,\lambda}$, the transformed of $N_{i,j,\lambda}$ by $M_jM_k$, and

* Jordan, § 218.

also $R_{i,j,\lambda}$, the transformed of $Q_{i,j,\lambda}$ by $M_iM_s$. Further, $J$ contains the substitutions

$$P_{i,j} = Q_{j,i,1}^{-1}\, Q_{i,j,1}\, Q_{j,i,1},$$
$$T_{1,\mu}T_{2,\mu} = M_1M_2P_{1,2}R_{1,2,\mu-1}N_{1,2,\mu}R_{1,2,\mu}{}^{-1}.$$

Having $T_{1,\mu}T_{2,\mu}$, $J$ contains its transformed by $P_{i,j}$ and hence contains the product

$$T_{1,\mu}T_{2,\mu}\cdot T_{2,\mu-1}T_{3,\mu-1}\cdot T_{3,\mu}T_{1,\mu} = T_{1,\mu^2}.$$

Thus if $m \geqq 3$, $J$ contains all the substitutions

$$P_{i,j}T_{i,\lambda}, \quad Q_{i,j,\lambda}, \quad R_{i,j,\lambda}, \quad N_{i,j,\lambda}, \quad M_iM_j.$$

4. THEOREM. *The group $J$ consists of the totality of substitutions* (1) *which satisfy the relations** (2) *and*

$$(3)\qquad \begin{cases} \sum_{j=1}^{m}\beta_j^{(i)}\delta_j^{(i)} = 0, \quad \sum_{j=1}^{m}\alpha_j^{(i)}\gamma_j^{(i)} = 0 \quad (i=1,\cdots,m) \\ \sum_{i,j}^{1\cdots m}\alpha_j^{(i)}\delta_j^{(i)} = m. \end{cases}$$

First, we prove that, if $\Sigma$ be a substitution (1) which satisfies the relations (2) and (3), then will also $M_rM_s\Sigma$ and $N_{r,s,\lambda}\Sigma$ satisfy them. It will then follow by induction that every substitution of $J$ satisfies the relations. The product $N_{r,s,\lambda}\Sigma$ when expressed in the form (1) has the coefficients

$$\bar{\alpha}_j^{(i)} = \alpha_j^{(i)}, \quad \bar{\beta}_j^{(i)} = \beta_j^{(i)} \quad (i,j=1,\cdots,m),$$
$$\bar{\gamma}_j^{(i)} = \gamma_j^{(i)}, \quad \bar{\delta}_j^{(i)} = \delta_j^{(i)} \quad (i,j=1,\cdots,m;\ j\neq r,s),$$
$$\bar{\gamma}_r^{(i)} = \gamma_r^{(i)} + \lambda\alpha_s^{(i)}, \quad \bar{\gamma}_s^{(i)} = \gamma_s^{(i)} + \lambda\alpha_r^{(i)}$$
$$\bar{\delta}_r^{(i)} = \delta_r^{(i)} + \lambda\beta_s^{(i)}, \quad \bar{\delta}_s^{(i)} = \delta_s^{(i)} + \lambda\beta_r^{(i)} \quad (i=1,\cdots,m)$$

Thus $$\sum_{j=1}^{m}\bar{\alpha}_j^{(i)}\bar{\gamma}_j^{(i)} = \sum_{j=1}^{m}\alpha_j^{(i)}\gamma_j^{(i)} + \alpha_r\cdot\lambda\alpha_s + \alpha_s\cdot\lambda\alpha_r = 0,$$

$$\sum_{i,j}^{1\cdots m}\bar{\alpha}_j^{(i)}\bar{\delta}_j^{(i)} = \sum_{i,j}^{1\cdots m}\alpha_j^{(i)}\delta_j^{(i)} + \lambda\sum_{i=1}^{m}(\alpha_r^{(i)}\beta_s^{(i)} + \alpha_s^{(i)}\beta_r^{(i)}) = m.$$

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* The conditions that a substitution (1) have the absolute invariant

$$\sum_{i=1}^{m}\xi_i\eta_i$$

in the $GF[2^n]$ are seen to be the relations (2) and (3), omitting the last one $\Sigma\,\alpha\delta = m$. The first hypoabelian group $G$ is thus completely defined by the invariant $\Sigma\,\xi_i\eta_i$.

The product $M_r\Sigma$ satisfies the first set of conditions (3), but not the last one, since its coefficients $\overline{\overline{a}}_i$ etc., give

$$\sum_{i,j}^{1\cdots m} \overline{\overline{a}}_j^{(i)}\overline{\overline{\delta}}_j^{(i)} = \sum_{i,j}^{1\cdots m} a_j^{(i)}\delta_j^{(i)} + \sum_{i=1}^{m} (\gamma_r^{(i)}\beta_r^{(i)} - a_r^{(i)}\delta_r^{(i)}) = m+1.$$

But the product $M_r M_s \Sigma$ evidently satisfies all of the conditions (3), the modulus being 2.

*Inversely, every substitution* (1) *satisfying the relations* (2) *and* (3) *belongs to the group J.*

We first find a substitution $S$ in $J$ which replaces $\xi_1$ by

$$f_1 \equiv \sum_{j=1}^{m} (a_j'\xi_j + \gamma_j'\eta_j),$$

where

$$\sum_{j=1}^{m} a_j'\gamma_j' = 0.$$

If $a_1' \neq 0$, we may take for $S$ the product

$$T_{1,a_1'}^{\cdot\cdot} Q_{1,2,a_2'} N_{1,2,\gamma_2'} \cdots Q_{1,m,a_m'} N_{1,m,\gamma_m'}.$$

If $a_1' = 0$, $\gamma_1' \neq 0$, we may choose for $S$

$$T_{1,\gamma_1'^{-1}} Q_{2,1,\gamma_2'} R_{1,2,a_2'} \cdots Q_{m,1,\gamma_m'} R_{1,m,a_m'} M_1 M_2.$$

Finally, if $a_j' = \gamma_j' = 0$ $(j = 1, \cdots, k-1)$, $a_k'$ or $\gamma_k' \neq 0$, there exists by the preceding cases a substitution $S'$ in the group $J$, replacing $\xi_k$ by $f_1$. We thus take $S = S' P_{1,k}$.

Thus, if $\Sigma$ denote the given substitution (1), we may set $\Sigma = S\Sigma'$, where $\Sigma'$ is a new substitution leaving $\xi_1$ fixed and, by the proof above, satisfying the relations (2) and (3). Let $\Sigma'$ replace $\eta_1$ by

$$f_1' = \sum_{j=1}^{m} (\beta_j'\xi_j + \delta_j'\eta_j),$$

where by $(2_1)$ and (3),

$$\delta_1' = 1, \quad \beta_1' + \beta_2'\delta_2' + \cdots + \beta_m'\delta_m' = 0. \tag{4}$$

A substitution in $J$ leaving $\xi_1$ fixed and replacing $\eta_1$ by $f_1'$ is given by

$$S' = R_{2,1,\beta_2'} Q_{2,1,\delta_2'} \cdots R_{m,1,\beta_m'} Q_{m,1,\delta_m'}.$$

Setting $\Sigma' = S'\Sigma_1$, $\Sigma_1$ will be a substitution leaving $\xi_1$ and $\eta_1$ fixed and satisfying the relations $(2_1)$ and (3). Hence it takes the form

$$\begin{cases} \xi_1' = \xi_1,\ \xi_i' = \sum_{j=2}^{m} (\alpha_j^{(i)}\xi_j + \gamma_j^{(i)}\eta_j) \\ \eta_1' = \eta_1,\ \eta_i' = \sum_{j=2}^{m} (\beta_j^{(i)}\xi_j + \delta_j^{(i)}\eta_j) \end{cases} \quad (i = 2, \cdots, m)$$

with conditions for the coefficients analogous to (2) and (3). Proceeding similarly with $\Sigma_1$, we find ultimately that

$$\Sigma = SS' \cdots S_{m-2}S_{m-2}'\, T_{m,\, a_m^{(m)}},$$

since a substitution altering only $\xi_m$ and $\eta_m$ and satisfying (2) and (3) is of the form $T_{m,\, a}$.

COROLLARY. The substitutions $M_i = (\xi_i\, \eta_i)$ do not belong to the Group $J$.

5. The order $\Omega_{m,\, n}$ of $J$ is readily determined. The number of distinct functions $f_1$ by which the substitutions of $J$ can replace $\xi_1$ is $P_{m,\, n} - 1$, if $P_{m,\, n}$ denotes the number of solutions of

$$\sum_{j=1}^{m} \alpha_j'\gamma_j' = 0.$$

But $$\alpha_1'\gamma_1' = \lambda, \quad \alpha_2'\gamma_2' + \cdots + \alpha_m'\gamma_m' = \lambda$$

gives $(2^{n+1} - 1)\, P_{m-1,\, n}$ sets of solutions when $\lambda = 0$, and $(2^n - 1)(2^{n(2m-2)} - P_{m-1,\, n})$ sets of solutions when $\lambda$ runs through the marks $+0$ of the $GF[2^n]$. Thus

$$P_{m,\, n} = 2^n P_{m-1,\, n} + (2^n - 1)2^{n(2m-2)}$$

By (4) the number of functions $f$ is $2^{n(2m-2)}$. Thus

$$\Omega_{m,\, n} = (P_{m,\, n} - 1)2^{2n(m-1)}\Omega_{m-1,\, n}.$$

Enumerating the substitutions of the form $T_{m,\, c}$, we have

$$\Omega_{1,\, n} = 2^n - 1 = \tfrac{1}{2}(P_{1,\, n} - 1).$$

Hence

$$\Omega_{m,\, n} = \tfrac{1}{2}(P_{m,\, n} - 1)2^{2n(m-1)}(P_{m-1,\, n} - 1)2^{2n(m-2)} \cdots (P_{1,\, n} - 1).$$

From the above investigation we derive the recursion formula

$$P_{s,\, n} - 1 = 2^n(P_{s-1,\, n} - 1) + (2^n - 1)\,(2^{2n(s-1)} + 1),$$

the initial term $P_{1,\, n} - 1$ being $2(2^n - 1)$. Then by induction we derive the result

$$P_{s,n} - 1 = (2^{ns} - 1)(2^{n(s-1)} + 1).$$

We thus obtain for the order of $J$ the simple formula

$$\Omega_{m,n} = (2^{nm} - 1)[(2^{2n(m-1)} - 1)2^{2n(m-1)}][(2^{2n(m-2)} - 1)2^{2n(m-2)}] \cdots [(2^{2n} - 1)2^{2n}].$$

*Simplicity of the group J, §§ 6–9.*

6. Let $I$ be an invariant sub-group of $J$ not the identity. By the proof in the *Quarterly Journal*, l. c., § 4, $I$ contains a substitution not the identity and replacing $\xi_1$ by $a\xi_1$. Further, if $m \geqq 3$, $I$ contains a substitution not the identity, leaving $\xi_1$ and $\eta_1$ fixed. The proof differs slightly from § 5 of the paper cited. Thus, for case (1), we may suppose $S_1$ to be commutative with every substitution of $J$ which leaves $\xi_1, \eta_1, \xi_2$ fixed. Equating the two values by which $S_1 R_{2,3,\lambda}$ and $R_{2,3,\lambda} S_1$ replace $\eta_2$ and the two by which they replace $\eta_3$, we have

$$\xi_3' = \delta_3'' \xi_3 + \delta_3'' \xi_3, \quad \xi_2' = \delta_2''' \xi_3 + \delta_3''' \xi_2.$$

Equating the two values by which $S_1 Q_{3,2,\lambda}$ and $Q_{3,2,\lambda} S_1$ replace $\xi_3$ and the two by which they replace $\eta_2$, we have

$$\xi_3' = a_3''' \xi_3 + \gamma_2''' \eta_3, \quad \eta_3' = \beta_3'' \xi_2 + \delta_2'' \eta_3.$$

Applying the conditions (2) and (3), $S_1$ takes the form

$$S_1 \begin{cases} \xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1 + \xi_2, \quad \eta_2' = a\xi_1 + \beta_2''\xi_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad + \eta_2 + \beta_2'''\xi_3 + a_3'''\eta_\beta + \cdots \\ \xi_2' = \xi_2, \quad \xi_3' = \xi_3 + a_2'''\xi_2, \quad \eta_3' = \eta_3 + \beta_3'''\xi_2, \cdots \end{cases}$$

where* $\beta_2'' = a_3'''\beta_2''' + \beta_4''\delta_4'' + \cdots + \beta_m''\delta_m''$.

The demonstration may now be completed as in *The Quarterly Journal*. In the proof of case (2) we need only make the trivial variation of replacing $M_2$ by $M_2 M_3$ which is permissible since $M_2$ leaves $\xi_2 + \eta_2$ unchanged.

7. Applying again the reasoning of §6 we conclude that, if $m \geqq 4$, $I$ contains a substitution leaving $\xi_1, \eta_1, \xi_2, \eta_2$ fixed; finally, that $I$ contains a substitution leaving fixed

$$\xi_i, \eta_i \quad (i = 1, \cdots, m-2).$$

Transforming it by $P_{1,m-1} P_{2,m}$ we reach a substitution $S$ of

* I do not find that $a_2'''$ must be zero, so that $S_1$ would reduce to $T_{1,a} R_{1,2,1}$ when $m = 3$, as stated by Jordan.

$I$ altering only $\xi_1, \eta_1, \xi_2, \eta_2$. Applying the conditions $(2_1)$ (3) we see that it has the form

$$S: \begin{cases} \xi_1' = \alpha_1'\xi_1 + \gamma_1'\eta_1 + \alpha_2'\xi_2 + \gamma_2'\eta, & \eta_1' = \beta_1'\xi_1 + \delta_1'\eta_1 \\ & \qquad + \beta_2'\xi_2 + \delta_2'\eta_2, \\ \xi_2' = \alpha_1''\xi_1 + \gamma_1''\eta_1 + \alpha_2''\xi_2 + \gamma_2''\eta_2, & \eta_2' = \beta_1''\xi_1 + \delta_1''\eta_1 \\ & \qquad + \beta_2''\xi_2 + \delta_2''\eta_2. \end{cases}$$

8. *If $m > 2$, the group $I$ contains a substitution $R_{\alpha,\lambda,\rho}$.*

Case I: $\gamma_1' \neq 0$. Transform $S$ by $T_{1,\gamma_1'^{-1}} R_{2,1,\alpha_2'} Q_{2,1,\gamma_2'}$ we reach a substitution $S_1$ in $I$ which replaces $\xi_1$ by $\gamma_1'^{-1}\eta_1$ but otherwise of the same form as $S$.

If $S_1$, be commutative with $M_2M_3$, it reduces* to

$$T_{1,\gamma_1'} M_1M_2Q_{2,1,\beta_2'} R_{1,2,\beta_2'}.$$

Case $(I_a)$: $\beta_2' = 0$. If $\gamma_1' = 1$, $I$ contains $M_1M_2$ and therefore its transformed by $N_{1,2,1}Q_{1,2,1}$, giving $R_{1,2,1}Q_{2,1,1}$ (Case III). If $\gamma_1' \neq 1$, $I$ contains

$$T_{1,\gamma_1'} M_1 M_2 \cdot P_{1,2}^{-1} T_{1,\gamma_1'} M_1M_2P_{1,2} = T_{1,\gamma_2'} T_{2,\gamma_2'^{-1}},$$

and hence its transformed by $M_2M_3$ giving $T_{1,\gamma_2'} T_{2,\gamma_1'}$. This in turn $R_{1,2,\lambda}$ transforms into $R_{1,2,\lambda(1+\gamma_1'^2)} T_{1,\gamma_1'} T_{2,\gamma_1'}$. Hence $I$ contains $R_{1,2,\lambda(1+\gamma_1'^2)} \neq 1$.

Case $(I_b)$: $\beta_2' \neq 0$. If $n > 1$, a mark $\rho \neq 1, \neq 0$ exists. The transformed of $S_1$ by $T_{2,\rho}$ will not be commutative with $M_2M_3$. If $n = 1$, $I$ contains

$$(M_1M_2Q_{2,1,1}R_{1,2,1})^{-1} P_{1,2} M_1M_2Q_{2,1,1}R_{1,2,1}P_{1,2} = P_{1,2}Q_{2,1,1},$$

when by Jordan, p. 204, ll. 15–17, $I$ contains $Q_{2,1,1}$.

If, however, $S_1$ be not commutative with $M_2M_3$, $I$ contains $S_1^{-1}M_2M_3S_1M_2M_3 \neq 1$ which leaves $\xi_1, \xi_2, \eta_2$ fixed (Case III.).

Case II. $\gamma_1' = 0$, $\alpha_2'$ and $\gamma_2'$ not both zero.

$I$ contains a substitution $T$ leaving $\xi_1$ fixed and replacing $\xi_2$ by

$$\alpha_1'\xi_1 + \gamma_1'\eta_1 + \alpha_2'\xi_2 + \gamma_2'\eta_2;$$

viz, if $\alpha_2' \neq 0$, $T \equiv T_{2,\alpha_2'} Q_{2,1,\alpha_1'}$;

while, for $\gamma_2' \neq 0$, $T \equiv M_2M_3T_{2,\gamma_2'} Q_{2,1,\alpha_1'}$.

Hence $S_1 \equiv T^{-1}ST$ replaces $\xi_1$ by $\xi_2$ and leaves $\xi_2, \eta_2$ fixed.

If $S_1$ be not commutative with $R_{1,2,\lambda}$, $I$ contains

$$S_1^{-1} R_{1,2,\lambda} S_1 R_{1,2,\lambda} \neq 1$$

which leaves $\xi_1$ fixed (Case III.).

*I do not find that $\beta_2'$ must $= 0$ as stated in Jordan, p. 204, l. 1.

If $S_1$ be commutative with $R_{1,2,\lambda}$ it reduces to

$$P_{1,2}Q_{2,1,\delta_1'}R_{1,2,\beta_1'}.$$

Then $I$ contains

$$P_{1,2}S_1\,P_{1,2}S_1^{-1} = Q_{2,1,\delta_1'}Q_{1,2,\delta_1'}.$$

But, when $\delta_1' = 0$, $I$ contains $P_{1,2}R_{1,2,\beta_1'}$ whose transformed by $Q_{1,2,1}$ leaves $\xi_1$ fixed (Case III.).

If $\delta_1' = 1$, $I$ contains

$$Q_{2,1,1}Q_{1,2,1} \equiv P_{1,2}Q_{2,1,1},$$

when, as in Case ($I_3$), $I$ contains $Q_{2,1,1}$.

If $\delta_1' \neq 0, \neq 1$, the transformed of $Q_{2,1,\delta_1'}Q_{1,2,\delta_1'}$ by $T_{1,\rho}$ gives $Q_{2,1,\delta\rho^{-1}}Q_{1,2,\delta\rho}$, so that $I$ contains

$$Q_{2,1,\delta\rho^{-1}}Q_{1,2,\delta\rho}\cdot Q_{2,1,\delta\sigma^{-1}}Q_{1,2,\delta\sigma},$$

which, for $\sigma = \rho(1+\delta^2) \neq 0$, reduces to

$$T_{1,1+\delta^2}T_{2,1+\delta^2}^{-1}.$$

Case III. $\gamma_1' = \alpha_2' = \gamma_2' = 0$.

We may verify in detail that $S$ reduces to

$$T_{1,\alpha_1'}R_{1,2,\beta_2'}Q_{2,1,\delta_2'}T_{2,\alpha_2''}.$$

Transforming this by $T_{2,\alpha_2''}^{-1}R_{1,2,\lambda}$, we obtain

$$R_{1,2,\lambda(1+\alpha_1'\alpha_2'')}T_{1,\alpha_1'}T_{2,\alpha_2''}R_{1,2,\beta_2'}Q_{2,1,\delta_2'}.$$

Thus if $\alpha_1'\alpha_2'' \neq 1$, $I$ contains $R_{1,2,\lambda(1+\alpha_1'\alpha_2'')} \neq 1$.

For $\alpha_2'' = \alpha_1'^{-1}$, I contains [writing $\alpha$ for $\alpha_1'$ and dropping affixes]

$$S = R_{1,2,\alpha\beta}Q_{2,1,\alpha\delta}T_{1,\alpha}T_{2,\alpha}^{-1},$$

$$P_{1,2}T_{2,\alpha}^{-1}ST_{2,\alpha}P_{1,2} = T_{1,\alpha^{-1}}T_{2,\alpha}R_{1,2,\beta}Q_{1,2,\delta}.$$

Hence $I$ contains their product

$$W \equiv R_{1,2,\beta(1+\alpha)}Q_{2,1,\alpha\delta}Q_{1,2,\delta}.$$

Transforming $W$ by $M_1\,M_3$ and the result by $P_{1,2}$, we obtain respectively,

$$Q_{1,2,\beta(1+\alpha)}N_{1,2,\alpha\delta}R_{1,2,\delta}, \quad Q_{2,1,\beta(1+\alpha)}N_{1,2,\alpha\delta}R_{1,2,\delta}.$$

Hence $I$ contains

$$Q_{2,1,\beta(1+\alpha)}\,Q_{1,2,\beta(1+\alpha)}.$$

Thus if $\beta(1+\alpha)+0$, $I$ contains some $R_{1,2,\rho}$ [see end of Case II]. If $\beta(1+\alpha)=0$, the transformed of $W$ by $T_{1,\alpha^{\frac{1}{2}}}$ gives

$$Q_{2,1,\delta\alpha^{\frac{1}{2}}}Q_{1,2,\delta\alpha^{\frac{1}{2}}},$$

so that we reach an $R_{1,2,\rho}$ unless $\delta=0$. But for $\delta=0$, $\beta(1+\alpha)=0$, $S$ reduces to $R_{1,2,\beta}$ or $T_{1,\alpha}\,T_{2,\alpha,-1}$ giving in either case an $R_{1,2,\rho}$.

9. Having a substitution of the form $R_{\alpha,\lambda,\rho}$, $I$ will contain its transform by $T_{\alpha,\lambda}$, giving $R_{\alpha,\lambda,\rho\lambda^{-1}}$. Thus $I$ contains $R_{\alpha,\lambda,\lambda}$ ($\lambda=$ arbitrary). Transforming it by $P_{\beta,\lambda}P_{\alpha,\epsilon}$ we reach $R_{\epsilon,\beta,\lambda}$. This $M_1M_2$ transforms into $Q_{\epsilon,\beta,\lambda}$, which in turn $M_\beta M_\epsilon$ transforms into $N_{\epsilon,\beta,\lambda}$. Finally, $Q_{1,2,1}N_{1,2,1}$ transforms $R_{1,2,1}\,Q_{2,1,1}$ into $M_1M_2$. Hence $I$ coincides with $J$, which is, therefore, a *simple group*.

10. For $m=2$, we may define the sub-group $J$ of index $2^n$ as follows:

$$J=\{M_1M_2,\ N_{1,2,\lambda},\ Q_{1,2,\lambda},\ T_{1,\lambda}\};$$

for it then contains also the substitutions

$$R_{1,2,\lambda}=M_1M_2N_{1,2,\lambda}M_1M_2,$$
$$Q_{2,1,1}=R_{1,2,1}Q_{1,2,1}N_{1,2,1}M_1M_2N_{1,2,1}Q_{1,2,1},$$
$$P_{1,2}=Q_{2,1,1}Q_{1,2,1}Q_{2,1,1}.$$

The order of $J$ is seen to be

$$\tfrac{1}{2}(P_{2,n}-1)\,2^{2n}\,(P_{1,n}-1)=\{2^n(2^{2n}-1)\}^2.$$

## Part II.—The Group $J_1$.* §§ 11–21.

11. Confining ourselves to the $GF[2^1]$, consider the first hypoabelian group $J'$ on the indices $x_2,\cdots,x_m$. Denote by $J_1$ the group obtained by extending $J'$ by the three substitutions $M_1M_2$, $L_1M_1$, $U$, viz.:

$$L_1M_1\colon\ x_1'=y_1,\quad y_1'=x_1+y_1.$$

$$U\colon\begin{cases}x_1'=x_2+y_2, & y_1'=y_1+y_2,\\ x_2'=x_1+x_2+y_2, & y_2'=x_1+y_1+x_2+y_2.\end{cases}$$

* $J_1$ is a sub-group of the Abelian Group (for $p=2$); for

$$U=P_{1,2}L_1Q_{2,1,1}L_2'.$$

12. THEOREM: *Every substitution* of $J_1$ is included among the linear substitutions*

$$(1)\qquad \begin{cases} x_i' = \sum_{j=1}^{m} (a_j^{(i)} x_j + c_j^{(i)} y_j) \\ y_i' = \sum_{j=1}^{m} (b_j^{(i)} x_j + d_j^{(i)} y_j) \end{cases} \qquad (i = 1 \cdots m)$$

*whose coefficients, taken modulo 2, satisfy the relations:*†

$$(5)\qquad \sum_{j=1}^{m} a_j^{(i)} c_j^{(i)} + a_1^{(i)} + c^{(i)} \equiv \sum_{j=1}^{m} b_j^{(i)} d_j^{(i)} + b_1^{(i)} + d_1^{(i)} \equiv \delta_{1i},$$

$$(i = 1, \cdots, m)$$

$$(6)\qquad \sum_{i,j}^{1 \cdots m} a_j^{(i)} d_j^{(i)} + a_1' + b_1' + c_1' + d_1' \equiv m,$$

*in addition to the relations* (2) [*when written in roman letters*]. Writing the relations (5) for the reciprocal of (1) we have

$$(5_1)\qquad \sum_{j=1}^{m} a_i^{(j)} b_i^{(j)} + a_1^{(j)} + b_1^{(j)} \equiv \sum_{j=1}^{m} c_i^{(j)} d_i^{(j)} + c_1^{(j)} + d_1^{(j)} \equiv \delta_{1j},$$

which must be a consequence of the relations (5) and (2).

Since the theorem is true for $L_1M_1$, $U$ and the substitutions of $J'$ [see § 4], it follows by induction if we prove that, when any substitution $\Sigma$ satisfies the above relations, $U\Sigma$ and $L_1M_1\Sigma$ also satify them.

Expressing $L_1M_1\Sigma$ in the form (1), it is seen to have the coefficients $\bar{a}_j^{(i)}$, $\bar{b}_j^{(i)}$, etc., where

$$\bar{a}_1^{(i)} = c_1^{(i)}, \quad \bar{c}_1^{(i)} = a_1^{(i)} + c_1^{(i)}, \quad \bar{b}_1^{(i)} = d_1^{(i)}, \quad \bar{d}_1^{(i)} = b_1^{(i)} + d_1^{(i)},$$
$$\bar{a}_j^{(i)} = a_j^{(i)}, \quad \bar{c}_j^{(i)} = c_j^{(i)}, \quad \bar{b}_j^{(i)} = b_j^{(i)}, \quad \bar{d}_j^{(i)} = d_j^{(i)} \quad (j = 2, \cdots, m).$$

The expression on the left of (6), when built in $\bar{a}_j^{(i)}$, $\bar{b}_j^{(i)}$, etc., reduces to $m$, on applying (6), in virtue of the relations

$$\sum_{i=1}^{m} c_1^{(i)} b_1^{(i)} = \sum_{i=1}^{m} a_1^{(i)} d_1^{(i)} + 1, \quad \sum_{i=1}^{m} c_1^{(i)} d_1^{(i)} + c_1' + d_1' = 1.$$

---

* The conditions that (1) shall leave

$$x_1 + y_1 + \sum_{i=1}^{m} x_i y_i$$

invariant modulo 2 are seen to be the relations (2) and (5). This invariant thus characterizes the second hypoabelian group $G_1$.

† Following Kronecker's notation,

$$\delta_{11} = 1, \quad \delta_{1i} = 0 (i + 1).$$

Similarly, $U\Sigma$ has the coefficients

$$\bar{a}_1^{(i)} = a_2^{(i)} + c_2^{(i)}, \quad \bar{c}_1^{(i)} = c_1^{(i)} + c_2^{(i)},$$
$$\bar{b}_1^{(i)} = b_2^{(i)} + d_2^{(i)}, \quad \bar{d}_1^{(i)} = d_1^{(i)} + d_2^{(i)},$$
$$\bar{a}_2^{(i)} = a_1^{(i)} + a_2^{(i)} + c_2^{(i)}, \quad \bar{c}_2^{(i)} = a_1^{(i)} + a_2^{(i)} + c_1^{(i)} + c_2^{(i)},$$
$$\bar{b}_2^{(i)} = b_1^{(i)} + b_2^{(i)} + d_2^{(i)}, \quad \bar{d}_2^{(i)} = b_1^{(i)} + b_2^{(i)} + d_1^{(i)} + d_2^{(i)},$$
$$\bar{a}_j^{(i)} = a_j^{(i)}, \quad \bar{b}_j^{(i)} = b_j^{(i)}, \quad \bar{c}_j^{(i)} = c_j^{(i)}, \quad \bar{d}_j^{(i)} = d_j^{(i)} \quad (j = 3, \cdots, m).$$

Thus

$$\sum_{i=1}^{m} (\bar{a}_1^{(i)}\bar{d}_1^{(i)} + \bar{a}_2^{(i)}\bar{d}_2^{(i)})$$

equals

$$\sum_{i=1}^{m} (a_1^{(i)}b_1^{(i)} + a_2^{(i)}b_2^{(i)} + a_1^{(i)}d_1^{(i)} + b_2^{(i)}c_2^{(i)})$$

$$+ \sum_{i=1}^{m} (a_1^{(i)}b_2^{(i)} + a_2^{(i)}b_1^{(i)}) + \sum_{i=1}^{m} (a_1^{(i)}d_2^{(i)} + b_1^{(i)}c_2^{(i)}).$$

The last two sums being zero, this reduces to

$$a_1' + b_1' + 1 + a_2' + b_2' + \sum_{i=1}^{m} (a_1^{(i)}d_1^{(i)} + a_2^{(i)}d_2^{(i)}) + 1.$$

Also

$$\bar{a}_1' + \bar{b}_1' + \bar{c}_1' + \bar{d}_1' = a_2' + c_1' + b_2' + d_1'.$$

Hence $U\Sigma$ satisfies the relation (6). A like result may be proven for the relations (2) and (5). Hence the substitutions (1) satisfying the relations (2), (5), (6) form a *group*. This abstract definition of our group is independent of Jordan's theory of "exposants d'échange," from which also the group property follows.

13. It is interesting to note that it is impossible to generalize to the $GF[2^n]$, $n > 1$, the group of substitutions (1) satisfying the relations (2), (5), (6).

Thus, the coefficients of $L_1 M_1 \Sigma$ satisfy (5) only if

$$(c_1^{(i)})^2 + c_1^{(i)} = 0, \quad (d_1^{(i)})^2 + d_1^{(i)} = 0,$$

*i. e.*, if $c_1^{(i)}$ and $d_1^{(i)}$ belong to the $GF[2^1]$. Similarly, by considering $M_1 L_1 \Sigma$, we find that $a_1^{(i)}$ and $b_1^{(i)}$ must be integers. Likewise the coefficients of $U\Sigma$ satisfy (5) only when

$$(a_1^{(i)})^2 + a_1^{(i)} + (a_2^{(i)})^2 + a_2^{(i)} = 0,$$

*i. e.*, if $a_2^{(i)}$ also belongs to the $GF[2^1]$. By considering $P_{23} U P_{23} \Sigma$, we find that $a_3^{(i)}$ must be integers, etc.

Further no generalization is gained by extending $J'$ by* $L_{1,\lambda}M_1$, since $L_{1,\lambda}'L_{1,1}L_{1,\lambda}'$ satisfies (5) only when $\lambda = 1$.

14. *Inversely, every linear substitution* (1), *satisfying the relations* (2), (5), (6), *belongs to the group* $J_1$. The proof varies only slightly from Jordan, §§ 279–281. At the end of § 279, we replace the three products by

$$S'UM_1M_2, \quad S'M_1M_2UM_1M_2, \quad L_1M_1S'UM_1M_2,$$

where $S'$ is the substitution in $J'$ which replaces $y_2$ by

$$\sum_{j=2}^{m}(a_j'x_j + c_j'y_j).$$

If $a_j' = c_j' = 0$ $(j = 2 \cdots m)$ we take for $S$ respectively:

$$L_1M_1,\ 1,\ L_1M_1 \cdot M_1M_2.$$

At the end of §280, we take for $S'$,

$$\bar{S}M_1L_1U^2 \text{ or } \bar{S}M_1M_2U^2,$$

according as $b_1' = 1$ or 0.

15. The order $\Omega_m'$ of the second hypoabelian group $J_1$ is not equal to the order $\Omega_m$ of the first hypoabelian group $J_{m-1}$ as stated by Jordan, § 279. In § 282 the reference should be to § 259 and not to § 260. Thus the number of solutions of

$$\sum_{j=2}^{m}a_j'c_j' + (a_1' + 1)(c_1' + 1) = 0$$

is $$P_m \equiv 2^{2m-1} + 2^{m-1}.$$

Hence $$\Omega_m' = 2P_mP_{m-1}\Omega_{m-1},$$

where $\Omega_{m-1} = (2^{m-1}-1)(2^{2m-4}-1)2^{2m-4}\cdots(2^2-1)2^2$.

The order of the group $J_1$ is thus

$$\Omega_m' \equiv (2^m+1)(2^{2m-2}-1)2^{2m-3}(2^{2m-4}-1)2^{2m-4}\cdots(2^2-1)2^2.$$

*Simplicity of* $J_1$, §§ 16–20.

16. In the main, I will follow Jordan's developments in §§ 283–86, but will replace §§ 287–89, in which a couple of small errors occur, by a simpler method, and finally wholly avoid the elaboration left to the reader in § 290.

In § 283 the treatment of the case $c_1' = 0$ will not verify.†

* For the notation see *The Quarterly Journal*, July, 1897.

† A simple correction suffices for Jordan's proof. Thus, $I$ contains $S^{-1}L_1'^{-1}SL_1'$ which leaves $x_1$ fixed and reduces to the identity only when $S$ itself leaves $x_1$ fixed.

If $a_j' = c_j' = 0 (j = 2 \cdots m)$, $S$ leaves $x_1$ fixed. In the contrary case, we may suppose, for example, that $a_2' = 1$. We may thus suppose that $a_3' = c_2'$; for, if not, the transformed of $S$ by $N_{2,3}$ replaces $x_1$ by

$$a_1'x_1 + a_2'x_2 + (c_2' + a_3')y_2 + a_3'x_3 + (c_3' + a_2')y_3 + \cdots$$

in which the coefficients of $x_2$ and $y_2$ are equal. Thus $I$ contains the substitution leaving $x_1$ fixed

$$S_1 = S^{-1}(M_1L_1)M_1M_2SM_1M_2(L_1M_1).$$

If $S_1$ is the identity, on comparing the values by which $S$ and $M_1L_1M_1M_2SM_1M_2L_1M_1$ replace $y_1$, we find that

$$x_1' = (b_2' + d_2')(x_2 + y_2) + d_1'x_1,$$

where by the relations (5),

$$(b_2' + d_2')^2 + d_1' = 1.$$

Since $S$ does not leave $x_1$ fixed, it replaces $x_1$ by $x_2 + y_2$. Hence $I$ contains $\overline{S}$, the transformed of $S$ by $N_{2,3}$, which replaces $x_1$ by $x_2 + y_2 + y_3$. Using this $\overline{S}$ in place of the former $S$, the product denoted by $S_1$ will not be the identity and will leave $x_1$ fixed.

17. We proceed as in §284, where the greater part of case 3° may be deleted. Indeed, the substitution $S_1$ given at the top of p. 210 is not hypoabelian; for a substitution replacing $y_2$ by $x_1 + y_2 + \beta(x_2 + x_3 + y_3)$ does not satisfy the relation (5),

$$0 = \sum_{j=1}^{m} \beta_j''\delta_j'' + \beta_1'' + \delta_1'' \equiv \beta + \beta^2 + 1 = 1.$$

18. As in §285, $I$ contains the substitution

$$A = [(P_{2,3}Q_{3,2})^{-1}(M_1M_2R_{2,3}Q_{3,2})^{-1}P_{2,3}Q_{3,2}(M_1M_2R_{2,3}Q_{3,2})]^2$$

which, on expansion, becomes

$$A: \begin{cases} x_2' = y_2, & y_2' = x_2 + y_2 + x_3 + y_3, \\ x_3' = y_2 + y_3, & y_3' = y_2 + x_3. \end{cases}$$

The substitutions $B \equiv B^{-1}$ and $C$ of Jordan, p. 210, are seen to belong to our group $J_1$. Thus $I$ contains

$$A^2 \cdot B^{-1}AB \equiv X: \begin{cases} x_1' = x_1, & y_1' = x_1 + y_1 + x_3, \\ x_2' = y_2, & y_2' = x_2, \\ x_3' = x_3, & y_3' = x_1 + x_3 + y_3. \end{cases}$$

Hence $I$ contains $M_1M_2XM_1M_2 \equiv M_1XM_1$ and, therefore, also $A^{-1}M_1M_2(M_1XM_1)M_2M_1A$ which gives the substitution

$$Y: \begin{cases} x_1' = x_1, \quad x_2' = x_2 + y_2 + x_3 + y_3, \quad x_3' = x_1 + x_2 + y_2, \\ y_1' = x_1 + y_1 + x_2 + y_2, \quad y_2' = x_1 + x_2 + y_2 + y_3, \\ y_3' = y_2 + x_3. \end{cases}$$

Hence $I$ contains the substitutions

$$Z \equiv (M_2M_3P_{2,3})^{-1}YM_2M_3P_{2,3} \cdot Y \equiv Q_{2,2}M_2M_3P_{2,3},$$

$$B \equiv C^{-1}ZC \cdot Y,$$

$$Z[(P_{2,3}Q_{2,3})^{-1}Z(P_{2,3}Q_{2,3}) \cdot B]^3 = Q_{2,2}(Q_{2,2}B)^3 = Q_{2,2},$$

as seen by a simple reduction using the fact that $B$ is commutative with $M_2M_3P_{2,3}$.

Containing $Q_{2,2}$, $I$ contains all the substitutions of $J'$ (see §§ 3 and 9).

19. The substitution $(UM_2)^3M_1M_2P_{2,3}$ transforms $N_{2,3}$ into $G \equiv G^{-1}$ (Jordan, bottom of p. 211).

The group $J_1$ contains the substitution *

$$U_1 \equiv U_1^{-1} \begin{cases} x_1' = y_2 + x_3 + y_3, \quad y_1' = x_2 + y_3, \\ x_2' = x_1 + y_1 + x_3 + y_3, \quad y_2' = x_1 + x_3 + y_3, \\ x_3' = y_1 + x_2 + y_2 + x_3, \quad y_3' = y_1 + x_2 + y_2 + y_3. \end{cases}$$

But $U_1$ transforms $M_2M_3$, $N_{2,3}R_{2,3}Q_{2,3}N_{2,3}$, and $N_{2,3}$ into respectively $L_1M_3$, $M_1M_2Q_{2,2}N_{2,3}$ and $F$ (bottom of p. 211). Hence $I$ contains $M_1M_2$ and, therefore, $M_1M_3$ and consequently the products:

$$L_1M_3 \cdot M_3M_1 = L_1M_1,$$

$$P_{2,3}FP_{2,3} \cdot M_1M_3 \cdot G \cdot M_1M_3 \cdot Q_{2,3}N_{2,3} = U.$$

Hence $I$ coincides with $J_1$, which is, therefore, *simple*.

20. *For $m = 2$, $J_1$ is of order 60 and is generated by* †

$$M_1M_2, \ L_1M_1, \ U, \ M_1UM_1.$$

We proceed as in Jordan, §279. If $a_2'c_2' = 1$, $U$ will replace $x_1$ by $f_1 = x_2 + y_2$. If $a_2'c_2' = 0$, we have three cases:

(1) $$a_2' = 0, \ c_2' = 1.$$

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* write $U_1$ for the French capital $U$ (p. 211, l. 13).

† We may drop $M_1UM_1$ from the list of generators; for,

$$U \cdot M_1UM_1 = L_1M_1 \cdot M_1M_2$$

Then will

$$UM_1M_2,\ M_1UM_1,\ \text{or}\ L_1M_1UM_1M_2.$$

replace $x_1$ by $f_1$, according as respectively,

$$a_1' = 0,\ c_1' = 1;\ a_1' = 1,\ c_1' = 0;\ \text{or}\ a_1' = c_1' = 1.$$

(2) $$a_2' = 1,\ c_2' = 0.$$

We choose respectively

$$M_2M_1 \cdot M_1UM_1,\ M_1M_2UM_1M_2,\ L_1M_1 \cdot M_2M \cdot M.UM_1.$$

(3) $$a_2' = c_2' = 0$$

We take respectively

$$L_1M_1,\ 1,\ L_1M_1 \cdot M_1M_2.$$

Continuing as in Jordan, § 280, we seek a substitution $S$, which replaces $y_1$ by $b_1'x_1 + y_1 + b_2'x_2 + d_2'y_2$, where $b_2'd_2' = 0$ without altering $x_1$.

(1) $$b_2' = d_2' = 0.$$

According as $b_1' = 0$ or 1, we take

$$S' = 1 \text{ or } M_1M_2 \cdot L_1M_1.$$

(2) $$b_2' = 1,\ d_2' = 0.$$

We take respectively

$$S' = (M_1UM_1)^2M_1M_2 \text{ or } M_1L_1U^2.$$

(3) $$b_2' = 0,\ d_2' = 1$$

We choose respectively

$$S' = M_1M_2U^2 \text{ or } M_1L_1 \cdot M_1M_2(M_1UM_1)^2M_1M_2.$$

Since no power of $M_1UM_1$ reduces to $U$, which is of period 5, $J_1$ contains more than one cyclical sub-group of order 5. Hence* $J_1$ is *simple*. To put it into the form of the icosahedral group, we may set

$$S_2 \equiv UM_1M_2,\ S_3 \equiv UM_1UM_2 = L_1M_1,\ S_5 = M_2U^{-1}M_2,$$

where $S_j$ is of period $j$. It follows that $S_2S_3S_5 = 1$.

21. We have reached the interesting result that the simple group $J_1$ on $m$ indices is obtained by extending the group

* Burnside: The Theory of Groups, pp. 107–8. The statements of Jordan § 291 are thus wholly wrong.

$J'$ on $m-1$ indices (a simple group if $m-1>2$) by the simple icosahedral group of degree 60.

The lowest orders of the simple groups $J$ and $J_1$ are seen to be as follows:

$$\Omega_2' = 60, \quad \Omega_3' = 2^6 \cdot 3^4 \cdot 5,$$
$$\Omega_4' = 2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17, \quad \Omega_5' = 2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17,$$
$$\Omega_{2,1} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 = \tfrac{1}{2}8!, \quad \Omega_{4,1} = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7,$$
$$\Omega_{5,1} = 2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31, \quad \Omega_{3,2} = 2^{12} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 17,$$
$$\Omega_{4,1} = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2, \quad \Omega_{3,3} = 2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73.$$

Denote by $(m, n, p)$ the order of the simple* group of linear fractional substitutions of determinant unity on $m-1$ indices in the $GF[p^n]$. We thus find

$$\Omega_{3,1} = (4, 1, 2) = (3, 2, 2), \quad \Omega_{3,3} = (4, 2, 2).$$

UNIVERSITY OF CALIFORNIA,
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## ON THE HAMILTON GROUPS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

ACCORDING to Dedekind a Hamilton group is a non-Abelian group all of whose subgroups are self-conjugate.† If the order of such a group is $p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots$ ($p_1, p_2, p_3, \cdots$ being prime numbers) it must be the direct product of its subgroups of orders $p_1^{a_1}, p_2^{a_2}, p_3^{a_3}, \cdots$ since each of these subgroups is self-conjugate and no two of them can have any common operator except identity.‡ Each of these subgroups is either Abelian or Hamiltonian. We proceed to prove that one of the given prime numbers must be 2 and that every subgroup whose order is a power of any other prime number must be Abelian.

Suppose that $G$ represents a Hamilton group of order $p^\alpha$, $p$ being an odd prime number. We may evidently select $a$ in such a manner that all the operators of $G$ whose orders

* Dickson: *Annals of Mathematics*, 1897, p. 136.
† *Mathematische Annalen*, vol. 48 (1897), p. 549.
‡ Cf. Dyck: *Mathematische Annalen*, vol. 22 (1883), p. 97.

do not exceed $p^{\alpha-1}$ $(\alpha > 1)$ are commutative to every operator of $G$ but that there are some operators of order $p^\alpha$ which do not possess this property. Let $s_1$ be one of the latter operators and let $s_2$ be an operator of $G$ that is not commutative to $s_1$, $S_1$ and $S_2$ representing the cyclical groups generated by these operators respectively. We shall first suppose that $s_2$ is also of order $p^\alpha$; $S_1$ and $S_2$ must contain at least $p$ common operators since $s_1$ and $s_2$ are not commutative.

$S_1$ and the subgroup of order $p^{\alpha-1}$ which is contained in $S_2$ generate an Abelian group whose order is equal to the order divided by $p$ of the group generated by $s_1$, $s_2$. This Abelian group must therefore contain all the operators of the latter group whose orders are less than $p^\alpha$. By hypothesis $s_1 s_2 s_1^{-1} = s_2^\beta$, where $\beta$ differs from unity. Hence

$$(s_2 s_1)^{p^{\alpha-1}} = s_2 s_2^\beta s_2^{\beta^2} s_2^{\beta^3} \cdots s_2^{\beta^{p^{\alpha-1}-1}} s_1^{p^{\alpha-1}} = s_2^{\frac{\beta^{p^{\alpha-1}}-1}{\beta-1}} s_1^{p^{\alpha-1}}$$

$\beta = xp^{\alpha-1} + 1$ where $x$ is one of the $p-1$ numbers, 1, 2, 3. $\cdots$, $p-1$.

Hence

$$\frac{\beta^{p^{\alpha-1}}-1}{\beta-1} = \frac{(1+xp^{\alpha-1})^{p^{\alpha-1}}-1}{xp^{\alpha-1}} \equiv p^{\alpha-1} \mod p^\alpha.$$

From this it follows that if $s_1$ remains fixed while $s_2$ takes all its possible values in $S_2$ the orders of some of the operators of the form $s_2 s_1$ must be less than $p^\alpha$. As this is impossible every operator of order $p^\alpha$ must be commutative to every other operator of this order.

It remains to consider the case where the order of $s_2$ exceeds $p^\alpha$. In this case $S_2$ and the subgroup of order $p^{\alpha-1}$ which is contained in $S_1$ generate an Abelian group. The order of each of the operators of the group generated by $s_1$, $s_2$ which is not contained in the Abelian group must exceed $p^{\alpha-1}$. Since the subgroup of order $p^\alpha$ which is contained in $S_2$ has at least $p$ operators in common with $S_1$ it follows that the product of $s_1$ into some operator of order $p^\alpha$ contained in $S_2$ must be of an order which does not exceed $p^{\alpha-1}$. Hence every operator of order $p^\alpha$ that is contained in $G$ is commutative to every operator of $G$ provided every operator of order $p^{\alpha-1}$ has this property, *i. e., there is no Hamilton group of order $p^\alpha$, $p$ being an odd prime number.* *

* This result could have been derived directly from the fact that the order of the commutator of two operators of a Hamilton group cannot exceed 2, which has been proved by Dedekind, loc. cit., p. 557. It seemed desirable to give an independent proof of it since this proof prepares the way for what follows.

*Hamilton groups of order* $2^n$.

By definition every operator of order 2 that is contained in such a group $H$ is self-conjugate. We proceed to prove that every operator of order 4 that is contained in $H$ is commutative to every operator whose order exceeds 4. If this is not the case we select an operator $s_1$ of order 4 that is not commutative to some operator $s_2$ of order $2^\beta$ $(\beta > 2)$ and we let $S_1$, $S_2$ represent the cyclical groups which these two operators generate respectively. $S_1$ and $S_2$ have two common operators, otherwise each operator of the one would be commutative to every operator of the other. Hence the order of the subgroup generated by $s_1$, $s_2$ is $2^{\beta+1}$. For convenience we shall call $S_2$ the head of this subgroup and its remaining $2^\beta$ substitutions we shall call its tail. Since the head is an Abelian group each of its operators must be either commutative to every operator of the tail or it must transform all the operators of the tail according to some regular substitution.

If $s_1$ and $s_2^{2^{\beta-2}}$ were commutative there would be some operators of order 2 in the tail of the given group. This is impossible since none of these operators is commutative to $s_2$. Hence $s_2$ must transform all the operators of this tail according to a regular substitution of order $2^{\beta-1}$. As this tail includes $s_1$ the order of this substitution cannot exceed 2. Hence this is impossible, *i. e.*, an operator of order 4 that is contained in $H$ must be commutative to every operator whose order exceeds 4 if such operators can occur at all.

Suppose $H$ contains two non-commutative operators $s_3$, $s_4$ of order $2^\beta$ $(\beta > 2)$ and that all the operators of this order are commutative to every operator of a lower order. $S_3$, $S_4$ will be used to represent the groups generated by $s_3$, $s_4$ respectively. These groups have at least two common operators. The Abelian group generated by $S_4$ and $s_3^2$ will be called the head of the group generated by $s_3$, $s_4$ and its remaining operators will be called the tail. This tail can evidently contain no operator whose order is less than $2^\beta$. We may suppose that $s_3 s_4 s_3^{-1} = s_4^\gamma$ where $\gamma$ differs from unity. Hence

$$(s_4 s_3)^{2^{\beta-1}} = s_4 s_4^\gamma s_4^{\gamma^2} \cdots s_4^{\gamma^{2^{\beta-1}-1}} s_3^{2^{\beta-1}} = s_4^{\frac{\gamma^{2^{\beta-1}}-1}{\gamma-1}} s_3^{2^{\beta-1}}$$

where

$$= 2^{\beta-1} + 1$$

$$\frac{\gamma^{2^{\beta-1}}-1}{\gamma-1}=\frac{(1+2^{\beta-1})^{2^{\beta-1}}-1}{2^{\beta-1}}\equiv 2^{\beta-1} \mod 2^{\beta}.$$

Hence the order of $s_4 s_5$ does not exceed $2^{\beta-1}$. As this operator occurs in the tail $S_5$ and $S_4$ must be commutative if they occur in $H$.

Suppose that $s_3$ is non-commutative to an operator $s_5$ whose order exceeds $2^\beta$. The Abelian group generated by $s_6$ and $s_3^2$ will be called the head of the group generated by $s_3$, $s_5$. Since $s_3$ is commutative to the subgroup of order $2^\beta$ contained in the group generated by $s_5$ and since the $2^{\beta-1}$ power of $s_5$ is contained in this subgroup the tail of the group generated by $s_3$, $s_5$ would contain operators whose order is less than $2^\beta$. Hence we see that if $H$ contains any operators whose order exceeds 4 each of these operators is commutative to every operator of $H$.

$H$ must, therefore, contain some operator $s_6$ of order 4 that is not commutative to each one of its operators. The operators which are commutative to $s_6$ will be called the head of $H$ and the rest of its operators will be called its tail. All the operators of the tail are of order 4 and the square of each one is equal to $s_6^2$, otherwise such an operator would be commutative to $s_6$*. The given head of $H$ contains some operators that are commutative to every operator of $H$. These form a subgroup which includes all the operators of order 2. We proceed to prove that this subgroup does not include any operator of order 4.

If it included an operator of order 4, the group of order 16 generated by $s_6$ and this operator of order 4 would contain operators of order 4 that are not commutative to the operators of the tail of $H$ and whose squares would differ from the squares of $s_6$. This is clearly impossible. Hence *$H$ does not include any operator whose order exceeds 4 nor does it contain any operator of order 4 that is commutative to all its operators.*

We have seen that all the operators in the given tail of $H$ have the same square. From what has just been proved each operator of order 4 may occur in a tail. Hence we see that *the square of every operator of order 4 that is contained in $H$ is equal to the square of a given one, i. e.*, all of these operators have the same square. This operator generates the commutator subgroup of $H$.

If we multiply $s_6$ by the subgroup which contains all the

* Dyck: loc. cit.

operators of order 2 we obtain an Abelian group which contains all the operators that are commutative to $s_6$. If this were not the case let $s_7$ be some other operator that is commutative to $s_6$. Since $s_7$ has the same square as $s_6$ its product into $s_6$ gives an operator of order 2. As this is impossible, the order of the given subgroup which contains all the operators of $H$ that are commutative to every one of its operators must be $2^{\alpha-2}$. Hence $H$ is completely determined when its order is given. If we add an operator of order 2 that is commutative to every operator of a Hamilton group of order $2^\alpha$ to such a group we clearly obtain a Hamilton group of order $2^{\alpha+1}$. Hence there is one and only one Hamilton group of order $2^\alpha$ whenever $\alpha$ exceeds 2. It is the direct product of the Abelian group of order $2^{\alpha-3}$ which contains no operator whose order exceeds 2 and the quaternion group.

*Summary.*

1. The order of every Hamilton group is even.

2. A Hamilton group of order $2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ ($p_1, p_2, \cdots$ being prime numbers) is the direct product of its subgroups of orders $2^\alpha, p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$. The first one of these subgroups is Hamiltonian and all the others are Abelian. The entire group may be represented as an intransitive substitution group in which each of these subgroups is represented by a distinct set of letters.

3. There is one and only one Hamilton group of order $2^\alpha$ ($\alpha > 2$). This group contains $2^{\alpha-2}$ operators of order 2. The rest of its operators are of order 4 and the squares of all of these operators are equal to each other. Every operator of order 4 is commutative to just one-half of all the operators of the group. The group is isomorphic to the 4-group with respect to the given subgroup of order $2^{\alpha-2}$.

4. If we multiply a Hamilton group of order $2^\alpha$ by any Abelian group of an odd order all of whose operators are commutative to every operator of the given Hamilton group, the product will be a Hamilton group and every possible Hamilton group can be constructed in this manner.

5. If two operators of a Hamilton group are not commutative they may be represented as two substitutions in which the cycles whose orders are divisible by 4 are of order 4, transform each other into their 3d powers, and belong to the subgroup of order $2^\alpha$. The letters which are found in these cycles do not occur in any of the remaining cycles of these two substitutions. The remaining cycles of these

substitutions are commutative. Hence the commutator of two such operators is of order 2.*

6. A Hamilton group of order $2^a$ contains $2^{2a-6}$ quaternion groups as subgroups. All of these have the commutator group of the entire group in common.†

CORNELL UNIVERSITY,
*April*, 1898.

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## NOTE ON THE INFINITESIMAL PROJECTIVE TRANSFORMATION.

BY PROFESSOR EDGAR ODELL LOVETT.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

IT is proposed here to find the form of the most general infinitesimal projective transformation‡ of ordinary space directly from its simplest characteristic geometric property. Geometrically, infinitesimal projective transformations of space are those infinitesimal point transformations which transform a plane into a plane, *i. e.*, which leave invariant the family of $\infty^2$ planes of ordinary space. Analytically, then, the most general infinitesimal projective transformation is the point transformation

$$Uf \equiv \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z} \quad (1)$$

which leaves invariant the partial differential equations

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* Cf. Dedekind: loc. cit.

† Cf. Miller: *Comptes Rendus*, vol. 126 (1898), pp. 1406–1408.

‡ In a note on the general projective transformation, *Annals of Mathematics*, vol. 10, No. 1, the forms of the finite projective transformations of ordinary space and those of $n$-dimensional space are found directly from the conditions for the invariance of the equations $y'' = 0$, $z'' = 0$, which expresses the geometric property that straight line is changed into straight line by these transformations. The form of the general infinitesimal projective transformation of ordinary space is deduced from the finite transformation by the method of Lie. In this derivation three steps are made to intervene, two of which are removed and the other replaced by a simpler one by the method of the present note: 1° two intersecting planes producing the straight line and its property of invariance; 2° the ordinary differential equations of the straight line and the conditions for their invariance; 3° the finite forms of the transformation.

$$\frac{\partial^2 z}{\partial x^2} = 0, \quad \frac{\partial^2 z}{\partial x \partial y} = 0, \quad \frac{\partial^2 z}{\partial y^2} = 0, \tag{2}$$

of all planes.

The infinitesimal transformation $Uf$ assigns to $x, y, z$ the respective increments

$$\delta x = \xi(x, y, z)\delta\varepsilon, \quad \delta y = \eta(x, y, z)\delta\varepsilon, \quad \delta z = \zeta(x, y, z)\delta\varepsilon, \tag{3}$$

where $\delta\varepsilon$ is an arbitrary infinitesimal of the first order.

In order to determine how $Uf$ changes the quantities

$$p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial^2 z}{\partial x^2}, \quad s \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad t \equiv \frac{\partial^2 z}{\partial y^2}, \tag{4}$$

*i. e.*, to find the increments $\delta p$, $\delta q$, $\delta r$, $\delta s$, $\delta t$ and thus determine what is known as the second extension of the point transformation $Uf$, namely

$$\begin{aligned} U'f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + \pi \frac{\partial f}{\partial p} + \varkappa \frac{\partial f}{\partial q} \\ + \rho \frac{\partial f}{\partial r} + \sigma \frac{\partial f}{\partial s} + \tau \frac{\partial f}{\partial t}, \end{aligned} \tag{5}$$

we proceed as follows :*

The variation of the identity

$$dz \equiv p dx + q dy \tag{6}$$

gives

$$d\delta z \equiv \delta p \cdot dx + \delta q \cdot dy + p d\delta x + q d\delta y, \tag{7}$$

by virtue of the fact that $d$ and $\delta$ are commutative. $\delta x$, $\delta y$, $\delta z$ are given functions of $x, y, z$ from (3). The identity (7) is to exist for all values of $dx$ and $dy$, hence it breaks up into two equations which determine $\delta p$ and $\delta q$.

Similarly the variation of the identities

$$dp \equiv r dx + s dy, \quad dq \equiv s dx + t dy, \tag{8}$$

yields

$$\left.\begin{aligned} d\delta p \equiv \delta r \cdot dx + \delta s \cdot dy + r d\delta x + s d\delta y. \\ d\delta q \equiv \delta s \cdot dx + \delta t \cdot dy + s d\delta x + t d\delta y. \end{aligned}\right\} \tag{9}$$

These two break up into four equations for $\delta r$, $\delta s$, $\delta t$. The computations of these increments result in

* See Lie: Vorlesungen über continuierliche Gruppen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1893, pp. 709–710.

$$\delta p = \pi\delta\varepsilon, \quad \delta q = \varkappa\delta\varepsilon, \quad \delta r = \rho\delta\varepsilon, \quad \delta s = \sigma\delta\varepsilon, \quad \delta t = \tau\delta\varepsilon, \tag{10}$$

where

$$\begin{aligned}
\varkappa &= \zeta_x + p\zeta_z - p(\xi_x + p\xi_z) - q(\eta_x + p\eta_z),\\
\varkappa &= \zeta_y + q\zeta_z - p(\xi_y + q\xi_z) - q(\eta_y + q\eta_z),\\
\rho &= \pi_x + p\pi_z + r\pi_p + s\pi_q - r(\xi_x + p\xi_z) - s(\eta_x + p\eta_z),
\end{aligned} \tag{11}$$

$$\left\{\begin{aligned}
\sigma &= \pi_y + q\pi_z + s\pi_p + t\pi_q - r(\xi_y + q\xi_z) - s(\eta_y + q\eta_z),\\
\sigma &= \varkappa_x + p\varkappa_z + r\varkappa_p + s\varkappa_q - s(\xi_x + p\xi_z) - t(\eta_x + p\eta_z),
\end{aligned}\right.$$

$$\tau = \varkappa_y + q\varkappa_z + s\varkappa_p + t\varkappa_q - s(\xi_y + q\xi_z) - t(\eta_y + q\eta_z).$$

The equations (2) are invariant by the transformation (1) when they are invariant by the twice extended transformation (5); they are invariant by the latter if

$$\delta r = 0, \quad \delta s = 0, \quad \delta t = 0 \tag{12}$$

for all values of $x, y, z, p, q$ when $r = 0, s = 0, t = 0$.

The complete computation of $\rho$, $\sigma$, and $\tau$ is very much abbreviated by omitting at once the terms involving $r$, $s$, $t$ and reckoning out only

$$\pi_x + p\pi_z, \quad \pi_y + q\pi_z, \quad \varkappa_y + q\varkappa_z. \tag{13}$$

The demand that these variations shall be zero for all values of $p$ and $q$ furnishes the following equations for the functions $\xi$, $\eta$, $\zeta$:

$$\xi_{yy} = 0, \quad \xi_{yz} = 0, \quad \xi_{zz} = 0; \tag{14}$$

$$\eta_{xx} = 0, \quad \eta_{xz} = 0, \quad \eta_{zz} = 0; \tag{15}$$

$$\zeta_{xx} = 0, \quad \zeta_{xy} = 0, \quad \zeta_{yy} = 0; \tag{16}$$

$$\begin{gathered}
\xi_{xx} - 2\zeta_{xz} = 0, \quad \eta_{yy} - 2\zeta_{yz} = 0, \quad \zeta_{xx} - 2\xi_{xz} = 0,\\
\zeta_{xx} - 2\eta_{yz} = 0;
\end{gathered} \tag{17}$$

$$\xi_{xy} - \zeta_{yz} = 0, \quad \eta_{xy} - \zeta_{xz} = 0, \quad \xi_{xx} - \eta_{yz} + \zeta_{xz} = 0. \tag{18}$$

These several sets lead to the following respectively

$$\xi = y\varphi_1(x) + \varphi_2(z, x) \equiv z\varphi_3(x) + \varphi_4(x, y); \tag{19}$$

$$\eta = x\varphi_5(y) + \varphi_6(y, z) \equiv z\varphi_7(y) + \varphi_8(x, y); \tag{20}$$

$$\zeta = x\varphi_9(z) + \varphi_{10}(y, z) \equiv y\varphi_{11}(z) + \varphi_{12}(z, x); \tag{21}$$

$$\begin{aligned} \xi_x - 2\zeta_x &= \psi_1(y, z), \quad \eta_y - 2\zeta_z = \psi_2(z, x), \\ \zeta_z - 2\xi_x &= \psi_3(x, y), \quad \zeta_z - 2\eta_y = \psi_4(x, y); \end{aligned} \tag{22}$$

$$\begin{aligned} \xi_x - \zeta_z &= \psi_5(z, x), \quad \eta_y - \zeta_z = \psi_6(y, z), \\ \xi_x - \eta_y + \zeta_z &= \psi_7(x, y). \end{aligned} \tag{23}$$

The equations (22) give

$$-3\xi_x = \psi_1(y, z) + 2\psi_3(x, y), \tag{24}$$

$$-3\zeta_z = 2\psi_1(y, z) + \psi_3(x, y). \tag{25}$$

The second form of $\xi$ in (19) and the first form of $\zeta$ in (21) give respectively

$$-3\xi_x = -3x\varphi_{9_x}(x) - 3\varphi_{4_x}(x, y), \tag{26}$$

$$-3\zeta_z = -3x\varphi_{9_x}(z) - 3\varphi_{10_z}(y, z). \tag{27}$$

Identifying (24) and (26), (25) and (27) respectively, we have

$$-3x\varphi_{9_x}(x) - 3\varphi_{4_x}(x, y) = \psi_1(y, z) + 2\psi_3(x, y), \tag{28}$$

$$-3x\varphi_{9_x}(z) - 3\varphi_{10_z}(y, z) = 2\psi_1(y, z) + \psi_3(x, y). \tag{29}$$

Since the left member of (28) is linear in $z$, $\psi_1(y, z)$ must be linear in $z$ and by (14) simultaneously linear in $y$ with no term of the form $yz$; hence

$$\psi_1(y, z) \equiv -3\alpha_1 y - 3\alpha_2 z - 3\alpha_3; \tag{30}$$

similarly from equations (29) and (16)

$$\psi_3(x, y) \equiv -3\beta_1 x - 3\beta_2 y - 3\beta_3.$$

Hence from (28)

$$\begin{aligned} \varphi_{9_x}(x) &= \alpha_2, \quad \varphi_3(x) = \alpha_2 x + \alpha_4; \\ \varphi_{4_x}(x, y) &= (\alpha_1 + 2\beta_2)y + 2\beta_1 x + (\alpha_3 + 2\beta_3), \\ \varphi_4(x, y) &= \alpha_5 xy + \alpha_6 x^2 + \alpha_7 x + \alpha_8 y + \alpha_9. \end{aligned} \tag{33}$$

Therefore finally, after making a convenient change of constants,

$$\xi(x, y, z) \equiv a + ex + hy + lz + \alpha x^2 + \beta xy + \gamma zx. \tag{34}$$

In a similar manner $\eta$ and $\zeta$ are found to have the following forms, where the choice of constants is determined by the equations (17) and (18)

$$\eta(x, y, z) \equiv b + fx + jy + mz + \alpha xy + \beta y^2 + \gamma yz, \quad (35)$$

$$\zeta(x, y, z) \equiv c + gx + ky + nz + \alpha zx + \beta yz + \gamma z^2. \quad (36)$$

Then the most general infinitesimal projective transformation of ordinary space has the form

$$\begin{aligned} Uf \equiv (a + ex + hy + lz + xu)\frac{\partial f}{\partial x} \\ + (b + fx + jy + mz + yu)\frac{\partial f}{\partial y} \\ + (c + gx + ky + nz + zu)\frac{\partial f}{\partial z}, \end{aligned} \quad (37)$$

where

$$u(x, y, z) = \alpha x + \beta y + \gamma z. \quad (38)$$

The finite forms of the transformation are found by means of Lie's theorem * by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1, z_1)} = \frac{dy_1}{\eta(x_1, y_1, z_1)} = \frac{dz_1}{\zeta(x_1, y_1, z_1)} = dt,$$

with the initial conditions

$$x_1 = x, \quad y_1 = y, \quad z_1 = z, \quad t = 0;$$

the integration yields

$$x_1 = L/P, \quad y_1 = M/P, \quad z_1 = N/P,$$

where $L$, $M$, $N$, $P$ are of the form

$$\alpha_i x + \beta_i y + \gamma_i z + \delta_i,$$

$\alpha_i$, $\beta_i$, $\gamma_i$, $\delta_i$ being constants and the index $i$ assuming the values 1, 2, 3, 4 successively.

PRINCETON, NEW JERSEY,
20 *April*, 1898.

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* Lie: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1891, Theorem 13, p. 218.

# INFINITESIMAL TRANSFORMATIONS OF CONCENTRIC CONICS.

BY PROFESSOR EDGAR ODELL LOVETT.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

A FAMILY of curves is invariant under the transformations of a continuous group of transformations when the family is invariant under the infinitesimal transformations which generate the group. A family is invariant under an infinitesimal transformation when the differential equation of the family admits of the infinitesimal transformation.

The criterion that a given differential equation of the $m$th order in $x$, $y$

$$\Phi(x, y, y', y'', \cdots, y^{(m)}) = 0, \tag{1}$$

admit of a known infinitesimal point transformation

$$Uf \equiv \xi(x, y) f \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y}, \tag{2}$$

is that

$$U^{(m)}\Phi \equiv 0, \tag{3}$$

where

$$U^{(m)}f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \cdots + \eta^{(m)} \frac{\partial f}{\partial y^{(m)}},$$

$$\eta' \equiv \frac{d\eta}{dx} - y' \frac{d\xi}{dx}, \cdots, \eta^{(m)} \equiv \frac{d\eta^{(m-1)}}{dx} - y^{(m)} \frac{d\xi}{dx},$$

give the $m$th extension of the original point transformation, $Uf$.

Conversely, if the differential equation is given and the infinitesimal transformation unknown, the condition (3) may be turned to account to find the forms of those infinitesimal point transformations of which the given differential equation admits. This converse problem is an integration problem not capable of general solution; in fact there are differential equations of the $m$th order which do not admit of infinitesimal point transformations.

If in particular the equation (1) is of the second order and can be put in the form

$$y'' - \omega(x, y, y') = 0, \tag{4}$$

the criterion becomes

$$\begin{aligned}(\eta_y - 2\xi_x - 3\xi_y y')\,\omega - \xi_{yy}y'^3 + (\eta_{yy} - 2\xi_{xy})y'^2 \\ + (2\eta_{xy} - \xi_{xx})y' + \eta_{xx} \qquad (5) \\ - \{\xi\omega_x + \eta\omega_y + [\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2]\omega_{y'}\} \equiv 0\end{aligned}$$

for all values of $x$, $y$, $y'$.

The differential equation of the first order

$$\omega(x, y, y') = 0$$

admits of the transformation (2) if the quantity within the braces in the preceding identity vanishes identically for all values of $x$, $y$, $y'$.*

In this note these well-known criteria are employed to determine those continuous groups of point transformations which leave the family of all concentric conics of the $xy$-plane invariant. The problem proves to possess an interesting solution.

The differential equation of a family of confocal conics is

$$(xy' - y)(yy' + x) - c^2y' = 0\,;\dagger \qquad (6)$$

from this the second order differential equation of all conics having the same centre is found to be

$$xyy'' + xy'^2 - yy' = 0, \qquad (7)$$

an equation which can be put immediately into the form (4),

$$y'' = \frac{y'}{x} - \frac{y'^2}{y}.$$

If the required infinitesimal transformation be taken in the form (2) the criterion (5) becomes

$$\begin{aligned}x^2y(y\xi_{yy} - \xi_y)y'^3 + x(2xy^2\xi_{xy} + 2y^2\xi_y - xy^2\eta_{yy} - xy\eta_y + x\eta)y'^2 \\ - y(2x^2y\eta_{xy} + 2x^2\eta_x - x^2y\xi_{xx} - xy\xi_x + y\xi)y' \qquad (8) \\ - xy^2(x\eta_{xx} - \eta_x) \equiv 0\,;\end{aligned}$$

* See theorems 21 and 35 of Lie's Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1891.

† This intermediate differential equation is introduced for use in a subsequent note. A neat derivation and integration of it are to be found in Jordan, Cours d'Analyse, 2d edition, vol. 1, No. 167; vol. 3, No. 36. It is obvious that the equation (7) is obtained in simpler manner directly from the equation $lx^2 + my^2 + n^2 = 0$.

whence the following equations of condition for determining the forms of the functions $\xi(x, y)$ and $\eta(x, y)$ :

$$y\xi_{yy} - \xi_y = 0, \tag{9}$$

$$2xy^2\xi_{xy} + 2y^2\xi_y - xy^2\eta_{yy} - xy\eta_y + x\eta = 0, \tag{10}$$

$$2x^2y\eta_{xy} + 2x^2\eta_x - x^2y\xi_{xx} - xy\xi_x + y\xi = 0, \tag{11}$$

$$x\eta_{xx} - \eta_x = 0. \tag{12}$$

The equations (9) and (12) demand that

$$\xi \equiv y^2X(x) + \varphi(x), \tag{13}$$

$$\eta \equiv x^2Y(y) + \psi(y) ; \tag{14}$$

these forms on being introduced into the equations (10) and (11) reduce them respectively to

$$y^2d(4xX) - x^2f(Y) - xf(\psi) = 0, \tag{15}$$

$$x^2d(4yY) - y^2f(X) - yf(\varphi) = 0, \tag{16}$$

where

$$f\{\rho(t)\} \equiv t^2\frac{d^2\rho}{dt^2} + t\frac{d\rho}{dt} - \rho. \tag{17}$$

The equations (15) and (16) give

$$X \equiv \frac{x^2}{16}\frac{f(Y)}{y^2} + \frac{x}{8}\frac{f(\psi)}{y^2}, \tag{18}$$

$$Y \equiv \frac{y^2}{16}\frac{f(X)}{x^2} + \frac{y}{8}\frac{f(\varphi)}{x^2}. \tag{19}$$

Then $X$ is a function of $x$ alone and $Y$ a function of $y$ alone if

$$\frac{f(X)}{x^2} = k, \quad \frac{f(Y)}{y^2} = l, \quad \frac{f(\varphi)}{x^2} = m, \quad \frac{f(\psi)}{y^2} = n, \tag{20}$$

where $k$, $l$, $m$ and $n$ are constants.

But it is readily seen that $k$ and $l$ must be zero ; for, taking the abridged forms

$$X_1(x) \equiv \frac{x^2}{16}\frac{f[Y_1(y)]}{y^2}, \tag{21}$$

$$Y_1(y) \equiv \frac{y^2}{16}\frac{f[X_1(x)]}{x^2}, \tag{22}$$

we have

$$Y_1(y) \equiv \frac{ky^3}{16}, \quad \frac{f[X_1(x)]}{x^3} \equiv k;$$

whence

$$X_1(x) \equiv \frac{kx^3}{8} + ax + \frac{b}{x}; \tag{23}$$

but

$$f[Y_1(y)] \equiv \frac{k}{2}y^3,$$

which gives

$$X_1(x) \equiv \frac{kx^3}{32}, \tag{24}$$

from (21). These two forms (23) and (24) for $X_1(x)$ are incompatible. In the same way we find inconsistent values for $l$. We have then incidentally the fact that it is impossible to find two functions $X_1(x)$ and $Y_1(y)$ satisfying the identities (21) and (22).

Hence the functions $X$, $Y$, $\varphi$, and $\psi$ are found by integrating the following ordinary linear differential equations of the second order:

$$f(X) = 0, \quad f(Y) = 0, \quad f(\varphi) = mx^3, \quad f(\psi) = ny^3;$$

this integration yields

$$X = ax + \frac{\gamma}{x}, \quad \varphi = \frac{m}{8}x^3 + \varkappa x + \frac{\lambda}{x};$$

$$Y = \beta y + \frac{\delta}{y}, \quad \psi = \frac{n}{8}y^3 + \mu y + \frac{\nu}{y};$$

the equations (9)–(12) or their equivalents (13)–(19) impose the following limitations on the constants:

$$a = \beta = \frac{m}{8} = \frac{n}{8}, \quad \gamma = \delta = 0;$$

whence finally

$$X = ax, \quad Y = ay;$$

$$\varphi = ax^3 + \varkappa x + \frac{\lambda}{x}, \quad \psi = ay^3 + \mu y + \frac{\nu}{y};$$

$$\xi = a(xy^2 + x^3) + \varkappa x + \frac{\lambda}{x}, \quad \eta = a(x^2y + y^3) + \mu y + \frac{\nu}{y}.$$

Hence the most general infinitesimal point transformation which leaves the family of all concentric conics invariant is

$$Uf \equiv \left\{ ax(x^2+y^2) + xx + \frac{\lambda}{x} \right\} \frac{\partial f}{\partial x}$$

$$+ \left\{ ay(x^2+y^2) + \mu y + \frac{\nu}{y} \right\} \frac{\partial f}{\partial y}.$$

PRINCETON, NEW JERSEY,
26 *March*, 1898.

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## A SOLUTION OF THE BIQUADRATIC BY BINOMIAL RESOLVENTS.

BY DR. GEORGE P. STARKWEATHER.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THE solution of a given equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

consists in making it depend upon a series of resolvent equations

$$R_1 = 0, \quad R_2 = 0, \cdots$$

whose solution may be effected by known methods. Thus the general quadratic is reduced to a binomial $x^2 = a$, the cubic to a quadratic and a binomial cubic, and the biquadratic to a cubic and three quadratic equations. In all the solutions of the biquadratic the writer has seen these resolvent equations are not binomial, although Galois' theory shows us that they may so be taken in an infinite variety of ways, according to the particular system of resolvent functions chosen. In selecting such a system it is of course desirable to find one which will give as simple results as possible, and after some trial the set employed in the following lines seemed to be the best. It is hoped this solution will be of interest from two points of view: 1° as giving a *new* solution of the biquadratic in which the roots are given explicitly, *i. e.*, ready for calculation; 2° as affording an interesting application of Galois' methods.

Let the coefficients of

$$x^4 + bx^3 + cx^2 + dx + e = 0$$

be independent variables. For the domain of rationality $R$, consisting of these and an imaginary cube root of unity, $a$, the group of this equation is the symmetric group $G_{24}$. Adjoining the square root of the discriminant $\Delta$ the Galoisian group becomes the alternate group $G_{12}$. Here

$$\Delta = 4^4 (I^3 - 27J^2),$$

where

$$I = e - \frac{bd}{4} + \frac{c^2}{12},$$

$$J = \frac{ce}{6} + \frac{bcd}{48} - \frac{d^2}{16} - \frac{b^2e}{16} - \frac{c^3}{216}.$$

An invariant subgroup of $G_{12}$ of index three is

$$G_4 = [1, (12)(34), (13)(24), (14)(23)].$$

Belonging to this is

$$\varphi_1 = x_1x_2 + x_3x_4,$$

which for the alternate group takes on the two other values

$$\varphi_2 = x_1x_3 + x_2x_4, \quad \varphi_3 = x_1x_4 + x_2x_3.$$

Form

$$\psi_1 = \varphi_1 + a\varphi_2 + a^2\varphi_3.$$

We then know that

$$\psi^3 - \psi_1^3 = 0$$

must be a rational equation, and in fact

$$\psi_1^3 = D + \frac{3\sqrt{3}}{2}\sqrt{-\Delta},$$

where

$$D = \tfrac{1}{2}(2c^3 - 9bcd + 27d^2 + 27b^2e - 72ce).$$

After adjoining $\psi_1$ the group of $f(x) = 0$ becomes $G_4$. The values of $\varphi_1$ and $\varphi_2$ are, therefore, rationally known and we have

$$\varphi_1 = \frac{\psi_1 + \dfrac{H}{\psi_1} + c}{3}, \qquad \varphi_2 = \frac{a^2\psi_1 + a\dfrac{H}{\psi_1} + c}{3},$$

where

$$H = c^3 - 3bd + 12e.$$

The values of $\varphi_1$ and $\varphi_2$ are obtained from the three equations

$$\psi_1 = \varphi_1 + \alpha\varphi_2 + \alpha^2\varphi_3,$$

$$\frac{H}{\psi_1} = \varphi_1 + \alpha^2\varphi_2 + \alpha\varphi_3,$$

$$c = \varphi_1 + \varphi_2 + \varphi_3.$$

An invariant subgroup of $G_4$ of primeindex is

$$G_2 = [1, (12)(34)].$$

Belonging to this is

$$\varphi_1' = x_1 + x_2$$

which for $G_4$ takes on the other value

$$\varphi_2' = x_3 + x_4.$$

Form $$\psi_1' = \varphi_1' - \varphi_2'.$$

Then $$\psi'^2 - \psi_1'^2 = 0$$

must be a rational equation, and in fact

$$\psi_1'^2 = b^2 - 4c + 4\varphi_1.$$

Adjoin $\psi_1'$ and the group becomes $G_2$.

$G_2$ has an invariant subgroup of prime index, the identical substitution, to which belongs

$$\varphi_1'' = x_1 + x_3.$$

For $G_2$ this takes on the other value

$$\varphi_2'' = x_2 + x_4.$$

As before, we form the equation

$$\psi''^2 - \psi_1''^2 = 0$$

where $$\psi_1'' = \varphi_1'' - \varphi_2''.$$

Then $$\psi_1''^2 = b^2 - 4c + 4\varphi_2.$$

Adjoining $\psi_1''$, the group becomes the identical substitution, and all rational functions of the roots are accordingly rationally known. In fact

$$x_1 = \frac{\psi_1' + \psi_1'' + \dfrac{K}{\psi_1'\psi_1''} - b}{4},$$

where

$$K = -b^3 + 4bc - 8d.$$

This is obtained from the four equations

$$x_1 + x_2 - x_3 - x_4 = \phi_1',$$

$$x_1 - x_2 + x_3 - x_4 = \phi_1'',$$

$$x_1 - x_2 - x_3 + x_4 = \frac{K}{\phi_1'\phi_1''},$$

$$x_1 + x_2 + x_3 + x_4 = -b.$$

$x_1$ is four-valued, for although $\phi_1'$ is six-valued three values of it are present, namely, $\phi_1'$, $\phi_1''$ and $\frac{K}{\phi_1'\phi_1''}$; and as the six values of $\phi_1'$ are of the form $\pm A$, $\pm B$, $\pm C$, if we make $\frac{K}{\phi_1'\phi_1''}$ negative, or $-A$ (supposing $K$ to be positive), we can only have

$$x_1 = \frac{-B + C - A - b}{4}$$

and

$$x_1 = \frac{B - C - A - b}{4};$$

while if we make $\frac{K}{\phi_1'\phi_1''}$ positive, or $+A$, we have

$$x_1 = \frac{B + C + A - b}{4}$$

and

$$x_1 = \frac{-B - C + A - b}{4}.$$

For convenience of reference the formulæ are appended in the order convenient for numerical application :

$$I = e - \frac{bd}{4} + \frac{c^2}{12},$$

$$J = \frac{ce}{6} + \frac{bcd}{48} - \frac{d^2}{16} - \frac{eb^2}{16} - \frac{c^3}{216},$$

$$\Delta = 4^4(I^3 - 27J^2),$$

$$D = \tfrac{1}{2}(2c^3 - 9bcd + 27d^2 + 27b^2e - 72ce),$$

$$\psi_1^3 = D + \frac{3\sqrt{3}}{2}\sqrt{-\Delta},$$

$$H = c^2 - 3bd + 12e,$$

$$\varphi_1 = \frac{\psi_1 + \dfrac{H}{\psi_1} + c}{3},$$

$$\varphi_2 = \frac{\alpha^2\psi_1 + \alpha\dfrac{H}{\psi_1} + c}{3},$$

$$\psi_1'^2 = b^2 - 4c + 4\varphi_1,$$

$$\psi_1''^2 = b^2 - 4c + 4\varphi_2,$$

$$K = -b^3 + 4bc - 8d,$$

$$x = \frac{\psi_1' + \psi_1'' + \dfrac{K}{\psi_1'\psi_1''} - b}{4}.$$

As an illustration let us apply these to the equation

$$x^4 - 2x^3 + x^2 + 2x - 2 = 0,$$

$$I = -\tfrac{11}{12}, \quad J = -\tfrac{37}{216}, \quad \Delta = -400, \quad D = 37,$$

$$\psi_1^3 = 88.96154, \quad \psi_1 = 4.46410.$$

(We take here the positive sign for $\sqrt{\Delta}$ and the real value of the cube root.)

$$H = -11, \quad \frac{H}{\psi_1} = -2.46410, \quad \varphi_1 = 1, \quad \varphi_2 = -2\sqrt{-1}$$

$$\left(\text{taking } \alpha = \frac{-1+\sqrt{-3}}{2}\right).$$

$$\psi_1'^2 = 4, \quad \psi_1''^2 = -8\sqrt{-1}, \quad \psi_1' = \pm 2,$$

$$\psi_1'' = \pm(-2 + 2\sqrt{-1}), \quad K = -16,$$

$$x_1 = -1, \quad x_2 = 1 + \sqrt{-1}, \quad x_3 = 1, \quad x_4 = 1 - \sqrt{-1}.$$

YALE UNIVERSITY,
*April*, 1898.

# NOTE ON SPECIAL REGULAR RETICULATIONS.

BY PROFESSOR ELLERY W. DAVIS.

As in Professor H. S White's first paper (BULLETIN, December, 1896) I write

$$\rho = r - 2, \quad \sigma = s - 2, \quad F = (4p - 4)(\rho + 2) / (\rho\sigma - 4).$$

With the exception of cases like $5_{14}$, $14_5$ and $6_{11}$, $11_6$ for $p = 9$, it appears from the tables and is a direct implication of Professor White's mode of building up the reticulations, that whenever for a given $p$ we have an $F$ less than its corresponding $V$, then there will also be found the same $F$ paired with a $V$ at most equal to it. But since $Fs = Vr$, this is the same as saying that when $\rho < \sigma$ for a given $F$, for the same $F$ we can also have $\rho \leq \sigma$.

Let $\sigma = \rho + K$. Then

$$K = \frac{1}{F\rho}\left[4(2p - 2 + F) + (4p - 4)\rho - F\rho^2\right]$$

and $\rho$ must divide $4(2p - 2 + F)$. If $2p - 2 + F = \delta_2\delta_3$, $\rho = \delta_1$ or $2\delta_1$ or $4\delta_1$ where $\delta_1$ is any factor of $2p - 2 + F$ including both unity and $2p - 2 + F$.

$$\rho = \delta_1 \text{ gives } K = K_1 = \frac{1}{F}(\delta_1 + 2)(2\delta_2 - F), \quad V = 2\delta_2.$$

$$\rho = 2\delta_1 \text{ gives } K = K_2 = \frac{2}{F}(\delta_1 + 1)(\delta_2 - F), \quad V = \delta_2.$$

$$\rho = 4\delta_1 \text{ gives } K = K_3 \frac{1}{F}(2\delta_1 + 1)(\delta_2 - 2F), \quad V = \delta_2/2.$$

When $F$, and therefore $\delta_2$, is odd, $K_3$ makes $V$ a fraction; otherwise we have our choice of the three $K$'s. Should, for a given $F$ and $p$, one of the $K$'s become a positive integer but none a negative integer, no matter what factor were taken of $2p - 2 + F$, an exceptional reticulation would be before us.

Plainly when $F = 1$ or 2 we can always have $K$ either zero or negative. When $F = 2$, however, and $p$ is odd, $F$ and $p - 1$ have the common factor 2 so that the reticulation is a derived one unless $V$ is odd. If we put $p = 2s + 1$, $\delta_2 = 1$, we find $K_3 = -4S - 1$, and $V = 1$ so that the re-

ticulation is special. Thus for $F=1$ or 2 it can never happen that there shall be a positive $K$ without a negative one.

Similarly, if $F=3$ there is no special reticulation for $p=3s+1$ while for $p=3s$ or $3s-1$ there is always one with a negative $K$.

When $F=4$ we can have $K_1=-p-3$ belonging to a special reticulation when $p$ is even, while for $p$ odd there is no special reticulation.

When $F=5$ we finally get exceptional reticulations provided $p=5s-1$ or $5s+2$ and the $s$ is rightly chosen. The simplest is that in Professor White's table, $5_{14}$, $14_5$.

Again, when $F=6$ there are sometimes exceptional reticulations for $p=6s+3$. The simplest is again one given in Professor White's tables $6_{11}$, $11_6$.

Other special reticulations occur for $F=7$. The simplest is $7_{13}$, $10_7$ for $p=10$.

In all attempts to realize these exceptional reticulations by construction I have failed. Nor do I see any way of proving that they cannot be constructed. This last once done would show Professor White's method to be exhaustive.

LINCOLN NEB.,
*April*, 1898.

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## LIMITATIONS OF GREEK ARITHMETIC.

BY MR. H. E. HAWKES.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

I PROPOSE to discuss in the present paper the number system of the Greeks, and to show how their arithmetical notions were limited by their geometrical symbolism. My argument is based chiefly on Euclid's Elements. This is not a serious limitation, for, firstly, the Elements give us practically all that Greek mathematicians knew on the subject, prior to 300 B. C., and, secondly, little was accomplished in this direction during the following three or four centuries. We may, therefore, consider Euclid's theory of number as representative.

I shall first attempt to show that Euclid naturally ex-

pressed and investigated problems relating to number and magnitude through the symbolism of lines. For example, he proves* in Book II, Proposition I, the general distributive law, which in his geometrical symbolism he expresses as follows:

"If there be two straight lines, and one of them be cut into any number of segments, the rectangle contained by the two straight lines equals the rectangles contained by the undivided line and the segments of the divided line." This in our literal symbolism would be expressed thus,

$$a(b + c + d + \cdots) = ab + ac + ad + \cdots$$

Later in the same book,† Proposition XI, Euclid shows how "to cut a given straight line so that the rectangle contained by the line and either segment equals the square on the remaining segment"—a problem which we should express thus—given a quantity $a$, find $x$ and $y$ such that they satisfy the equations

$$x + y = a, \quad ax = y^2.$$

It is a remarkable fact that the rest of the book is chiefly occupied by proofs of particular cases of the distributive law. On account of his unwieldy symbolism Euclid does not recognize the fact that he has already proved the general case. Though he does not expressly make such a statement, it is very probable that Euclid intended the problems of this book to be true for magnitudes in general, for which the lines are merely the symbols.*

In Book V, Euclid considers the ratio of homogeneous magnitudes, and in the following books he applies the theorems there proven to particular kinds of magnitudes, as surfaces, solids, and numbers.† Throughout Book V, the symbol by which he expresses these general magnitudes is the line. How cumbersome his notation is, can be gathered from the fact that proofs which consume thirty-four pages of Heiberg's edition of the Elements would occupy only three or four expressed in our literal symbolism.

In Book X the problem of incommensurability is investigated, a question which does not appear in the Elements up to that point owing to the power of Euclid's theory of ratio. His investigations in this book on the incommensurability

* Euclid's Elementa, edited by I. L. Heiberg, Leipzig, 1883, vol. 1, p. 118.

† Heiberg: loc. cit., vol. 1, p. 153.

* Cf. M. Cantor: Vorlesungen über die Geschichte der Mathematik, vol. 1, p. 249; also J. Gow: Short history of Greek mathematics, Cambridge, 1884, p. 72.

† Heiberg: loc. cit., vol. 4, p. 138; vol. 4, p. 169; vol. 2, p. 334.

of his magnitude symbols are precisely analogous to the work of Dedekind, Cantor, and Weierstrass on irrationalities. Here he considers all lines of the type

$$A = \sqrt{\sqrt{x} \pm \sqrt{y}}$$

where $x$ and $y$ are commensurable. He finds that there are twenty-five distinct or independent classes of such lines.* He also shows how to find individuals of each class, a process which leads him to the solution of certain equations of the fourth degree. This problem he attacks and solves with his line notation.† It is certainly very wonderful that Euclid could exhaust the subject of incommensurable lines of this type by means of his clumsy notation, but in this very book the weakness of the line symbolism shows itself most clearly in the limitations which it imposed on his concept of incommensurability. He understood the kind of incommensurability which we call square root, for he could construct a square equivalent to a given rectangle whose sides were commensurable. He also arrived at what we know as the fourth root of a quantity by a similar construction. His symbolism would allow him to consider the eighth root, the sixteenth root, etc., but that is as far as his line notation would permit him to go in space as we perceive it. We see that the choice of a magnitude symbol which depends on our space intuitions is a serious limitation.

If we ask how far the Greeks went toward arranging their line symbols into a dense multiplicity or a continuous system, we must answer that they took no steps at all. Euclid assumes that the sum of two lines is a line, that a short line taken from a long line leaves a line. The difficulty which arises if we wish to take a long line from a short one never seems to trouble him. The product of two lines is a rectangle, of three lines a solid. He does not concern himself with the process of division. He considers incommensurable lines, as we have seen, but takes no steps toward arranging his symbols in order of magnitude much less as a dense multiplicity or a continuum.

Let us now consider the effect of the line notation on the development of the number system of the Greeks. It will be evident that Euclid did not consider the line as merely a con-

* See De Morgan's article in the Penny Encyclopedia: "Irrationality."

† S. A. Christensen: "Über Gleichungen vierten Grades im zehnten Buch der Elemente Euclid's." *Zeitschrift für Mathematik und Physik, Historisch-literarische Abteilung*, 1889, p. 201.

venient graphical representation for number, but that he regarded the line as the fundamental symbol for any magnitude of which number was a special case. I consider in this paper only the theoretical arithmetic or *Arithmetica*, not the practical arithmetic or *Logistica* of the shopkeeper or the surveyor. The two were entirely distinct in the mind of the Greek.

We cannot find a time when the Greeks were not familiar with the smaller positive integers and did not have names for them. One would say that these would be the most natural symbols for mathematical magnitudes. It may be, however,* that the discovery of the incommensurable by Pythagoras destroyed the confidence of the Greeks in the integers as magnitude symbols. The finding of a line which could not be expressed in terms of a given line by means of integers may well have suggested to them that the line would be capable of expressing more magnitudes than the integers and thus give rise to a richer multiplicity. Thus they were led to leave reasoning by means of numerical notation to the shopkeeper and practical mathematician. The pure mathematician considered the line as the only safe notation. The separation of geometry from analysis was as complete as Weierstrass could wish, but with the opposite order of importance.

Consider now the Greek conception of operations on integers and note the constantly narrowing effect of the line notation.

Euclid assumes that two numbers added make a number† as is evident from the fact that a line plus a line generates a new line. He also assumes that a number taken from a number leaves a number.‡ He infers that we always take the less from the greater and never does he betray a suspicion that a case may arise where the greater might be subtracted from the less. When he solves equations by his line notation§ he finds only positive roots. Even Diophantos (250? A.D.)‖ rejects negative solutions to his problems, and nowhere to the end of Greek mathematics do we find a negative number.¶ It is easy to see that taking a long line from a short one would seem so absurd that they would never think of the operation of taking a greater number

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* H. B. Fine: Number System of Algebra, Boston, 1890, p. 97.
† Heiberg: loc. cit., vol. 2, p. 393 ff.
‡ Vol. 2, p. 395
§ Vol. 1, p. 153.
‖ T. L. Heath: Diophantos of Alexandria, Cambridge, 1885, pp. 82, 201, 207.
¶ M. Cantor: Geschichte der Mathematik, vol. 1, p. 401.

from a less one, since from their point of view the latter would be a special case of the former. Thus the line symbolism limited the Greek number system by one-half in depriving it of negative numbers.

From Plato* we learn that a number made up of the product of two unequal factors is an oblong number; if the factors are equal it is a square number. Similarly they had solid numbers and cube numbers. Suppose however they wished to express a product composed of four factors. If the numbers were to represent lines, the product would carry them into four dimensions. This was a difficulty which Euclid saw and could not meet. For whenever he considered the product of two numbers he first postulated the existence of that product.† For instance, Book IX, Proposition XXVIII: "If an odd number multiplying an even number *make any number*, the product will be even." Again Book IX, Proposition I: "If two similar plane numbers multiplying each other *make any number*, the number produced by them will be a square." Plainly he was not sure of the existence of the product, if the numbers were such as would involve more than three dimensions for their geometrical expression.

Thus the line notation imposes a further bond on the Greek number system in that it makes the consideration of more than three factors a peculiarly delicate matter. This very meager number system which we have sketched, viz., the positive integers which can be considered as the product of not more than three factors, is the end of the growth of the Greek number system until the time of Diophantos who introduced the rational fraction.‡ "No case of simple division occurs in Greek arithmetical literature."§ The fraction was to the Greeks the ratio of two numbers and did *not* generate a new number. Euclid in Book V. considers the ratio of homogeneous quantities, but he does not conceive of these ratios as quantities at all. They were not subject to the axioms to which all quantities are subject, *e. g.*, in Book V, Proposition XI,‖ he proves, "Ratios which are equal to the same ratio are equal to each other." It is impossible to tell how far the fact that division does not occur as an explicit process is due to the lack of any well-defined similar process on lines; but, judging from the

* Plato: Theaetetus, translated by Jowett, vol. 3, p. 347.
† Heiberg: loc. cit., vol. 2. pp. 340–347, 396 ff.
‡ M. Cantor: Geschichte der Mathematik, vol. 1, pp. 159, 405.
§ J. Gow: Short history of Greek mathematics, p. 51.
‖ Heiberg: loc. cit., vol. 2, p. 35.

close dependence of the operations of subtraction and multiplication of numbers on the same operations for lines, it would seem that there must be some connection.

It is evident that the Greeks were even farther from a continuum of numbers than they were from a continuum of their line symbols. For no number in their system existed which expressed their incommensurable lines, not to mention the countless kinds of incommensurability of which they knew nothing.

These considerations indicate that Greek mathematics rested on a very narrow basis so long as it clung to its line notation. The sense of rigor, as shown by postulating the existence of a product of two factors certainly would not allow them to assume a continuous system, as less careful mathematicians have done. This line notation did not admit of sufficient expansion to allow them to establish on *that* such a system. Thus, until the foundation of their mathematical science was utterly changed, an advance to algebra and calculus was impossible.

YALE UNIVERSITY,
*April*, 1898.

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## MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.

BY PROFESSOR JAMES PIERPONT.

IN treating the theory of maxima and minima in my lectures this year I have been astonished to find that the presentation of this theory in all English and American textbooks on the calculus which I could consult was false. That the older editions of such standard treatises as Todhunter, Williamson, and Byerly should be wrong in this particular was not astonishing since it was only in 1884 that Peano in his critical notes to the Calcolo Differenziale of Genocchi called attention to the point in question. Since then L. Scheeffers,* O. Stolz,† and von Dantscher‡ have devoted memoirs to this interesting but difficult subject and their results have found a place in the new edition of the Cours d'Analyse of C. Jordan and the Grundzüge of O. Stolz.

* *Mathematische Annalen*, vol. 35, p. 541.
† *Sitzungsberichte Vienna Academy*, 1890 (June).
‡ *Annalen*, vol. 42, p. 89.

Our English and American authors seem however to be totally ignorant of these facts. In the latest editions of the treatises just mentioned* as well as in the countless new works on the calculus that are constantly appearing we find the same incorrect reasoning repeated with an innocent ignorance that must be highly amusing to continental mathematicians. Yet this deplorable state of affairs would not have called forth the present note if the first calculus† written in the English language with the avowed purpose of presenting the subject in accordance with modern standards of rigor, did not employ the traditional incorrect treatment. That the author has not been led to give an incorrect criterion is not due to his method but because he gives no criterion at all for the critical case, for which he contents himself by remarking that *the expansion must be continued further.*‡ That this manner of reasoning is not confined to England and America and may be adopted by an otherwise careful writer is illustrated by the excellent work of Demartres, Cours d'Analyse,§ written in thorough harmony with the modern spirit of rigor. Here the reasoning is carried to its logical consequence for the critical case just alluded to and a false criterion is arrived at.

In the light of these facts I think it worth while to call the attention of American mathematicians to this matter once more. The point in question is very simple. To determine whether $f(x_1, x_2, \cdots, x_n)$ has an extreme at the point $P = (a_1, a_2, \cdots, a_n)$ we develop it into Taylor's series, this being possible, and get

$$\Delta = f(x_1, \cdots, x_n) - f(a_1, \cdots, a_n) = H_1 + H_2 + H_3 + \cdots$$

If $P$ is an extreme $\Delta$ must have one sign for all points in its vicinity.‖ To deduce a criterion, the assertion is now made in the works we are criticising that the sign of $\Delta$ depends upon the sign of $H_m$, that being the first derivative which does not vanish identically, while for the particular points for which $H_m$ may vanish the sign of $\Delta$ depends upon the sign of the first derivative which does not vanish for

* For example, the 8th edition of Williamson's Differential Calculus, dated 1895.

† Infinitesimal Calculus by Horace Lamb, Cambridge University Press, 1897.

‡ Loc. cit., pp. 596–597.

§ Hermann, Paris, 1892–96; cf. vol. 1, p. 72–73.

‖ The case when $\Delta$ may be zero for certain points is not considered in our text-books.

these points. This, of course, is generally wrong. The reason why so many mathematicians have fallen into this error seems to be this: For small values of the increments $h_1, h_2, \cdots, h_n$, $H_m$ is an infinitesimal of the $m$th order, while all following terms being of higher order would be generally infinitely small in comparison with $H_m$. This is also the probable reason why some writers set

$$\Delta = H_m + H_{m+1}$$

all other terms being simply neglected. For a single variable this is quite true, and without further thought it has been assumed for $n$ variables. The fact that $H_m$ can vanish for all points of a right line in any domain however small about $P$ while the higher terms may not, a fact which at once marks a difference between functions of one and of several variables, has not made mathematicians more cautious in making the above assumption.

To return, this position being once taken, the conclusion is correctly drawn that if $H_m$ is a definite form the function has an extreme at $P$, while if $H_m$ is indefinite no extreme exists. A peculiar mistake is now made by some authors by neglecting the fact that still a third case is possible, viz., when $H_m$ is semi-definite. So Williamson, who in treating the case for two variables $x$, $y$ states that an extreme exists if only the determinant

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \gtrless 0$$

at $P$, which of course is wrong. In passing I wish to remark for the benefit of the non-professional mathematician that the article on the Infinitesimal Calculus which this author contributed to the last edition of the Encyclopedia Britannica, vol. 13, p. 24, contains the same mistake. To show how the case when $H_m$ is a semi-definite form is treated we consider only two variables because a higher number is not usually given. Suppose then that

$$f(a + h, b + k) - f(a, b) = \Delta = H_2 + H_3 + \cdots$$

and

$$D = 0.$$

Then for all points on the right line $L$

$$h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial x \partial y} = 0$$

passing through $P$, $H_2 = 0$. This being so the reasoning already characterized as false shows that for all points on $L$ we must have $H_2 = 0$ for an extreme, so that the sign of $\Delta$ for these points depends upon that of $H_4$. Hence the conclusion: $f(x, y)$ has an extreme at $P$, if $H_4$ for points on $L$ has the sign of $\frac{\partial^2 f}{\partial x^2}$ at $P$.

*To restate the criticism:* the fundamental error is the assumption that when $H_m$ does not vanish the value of the remaining terms

$$R = H_{m+1} + H_{m+2} + \cdots$$

is small in comparison with $H_m$ so that the sign of

$$\Delta = H_m + R$$

depends upon that of $H_m$.

To illustrate the various points, consider

$$f(x, y) = x^2 - 6xy^2 + 8y^4$$

$$= H_2 + H_3 + H_4$$

for the point $P = (0, 0)$. This is a particular case of Peano's classic example. Here in the first place $D = 0$, so that according to Williamson we have an extreme and in fact a minimum. Let us apply now Demartres' criterion. The line $L$ is here $h = 0$; for these points $H_3 = 0$ while the sign of $H_4$ being positive is the same as that of

$$\frac{\partial^2 f}{\partial x^2} = +2.$$

Hence we would have also here a minimum. The fact is that $P$ is neither maximum or minimum, as is at once seen in letting $x$, $y$ move along the parabola

$$h = mk^2.$$

Then $$\Delta = k^4 (m - 2)(m - 4),$$

which shows that for no domain however small about $P$, has $\Delta$ always one sign, since by varying $m$, that is for different parabolas, the sign of $\Delta$ is different.

This example illustrates very well the fundamental error stated above. According to this, for those points for which

$H_2 \neq 0$ the value of $R = H_3 + H_4$ is small compared with $H_2$. That this is not so is easily seen. For let $x$, $y$ move along the parabola

$$k^2 = h.$$

Then $H_2 = h^2$ while $R = 2h^2$, *i. e.*, $R$ is twice as large as $H_2$. Still the sign of $\Delta$ is that of $H_2$.

For the parabola

$$3k^2 = h$$

$H_2 = h^2$ while $R = -\frac{1}{9}h^2$, so that the sign of $\Delta$ is that of $R$ and not that of $H_2$.

For the parabola

$$-10k^2 = h,$$

$$H_2 = h^2, \quad R = \tfrac{6}{100}h^2,$$

which is the only one of these three examples in which $R < H_2$.

In terminating this note I wish to remark that in the general case where no attempt is made to use Taylor's series, but where we suppose only the first derivatives to exist at the point in question, $P$, the oversight is frequently made that the extremes of $f(x_1, x_2, \cdots, x_n)$ are stated to be the points for which

$$f_{x_i}'(x_1, x_2, \cdots, x_n) \quad (i = 1, 2, \cdots, n)$$

vanish. This of course is not so in general, since the value of a partial derivative at a point may be quite different from the value obtained by taking $x_1, x_2, \cdots, x_n$ arbitrary, computing the general expression of $f_{x_i}'(x_1, x_2, \cdots, x_n)$ and then setting in this $x_1 = a_1$, $x_2 = a_2$, $\cdots$, $x_n = a_n$.

YALE UNIVERSITY,
*June*, 1898.

# ON THE INTERSECTIONS OF PLANE CURVES.

BY F. S. MACAULAY, M.A.

In a review of the theory of the "Intersections of plane curves" in the March number of the BULLETIN (pp. 260–273), Professor Charlotte A. Scott has included a full and appreciative criticism of my paper on "Point-Groups in relation to curves" (*Proceedings of the London Mathematical Society* vol. 26 (1895), pp. 495–544). I am exceptionally fortunate in having my work in this subject so clearly described and explained; and I hope I may be allowed to discuss further some interesting points raised in Miss Scott's paper.

For the sake of clearness it may be well to repeat what is meant by excess and defect. If a $C_6$ and $C_7$ are drawn through 4 points on a straight line they intersect again in 38 points. These 38 points are such that a $C_8$ through 35 of them necessarily passes through the remaining 3, and a $C_9$ through 37 of them necessarily passes through the last, so that the 38 points supply only 35 independent conditions for a $C_8$, and 37 for a $C_9$; these properties are expressed by saying that the 8-ic excess of the group of 38 points is 3, and the 9-ic excess is 1. So, in general the $n$-ic excess $r_n$ of a group of $N$ points is the excess of $N$ over the number of independent conditions that the point group $N$ supplies for $n$-ics. This number of conditions is therefore $N - r_n$. So also the $n$-ic defect $q_n$ of the same point group $N$ is the number of independent conditions by which the group falls short in determining an $n$-ic; in other words, it is the degree of freedom of the general $n$-ic through the point group $N$. Hence the formula

$$N - r_n + q_n = \tfrac{1}{2}n(n+3). \qquad (1)$$

Of course $r_n$ may be zero; but it is important to bear in mind, if $N$ is a point group derived in some way from the intersection of curves, that $r_n$ is just as likely to be greater than zero as to be zero.

The terms excess and defect are nearly equivalent to the terms suggested by Cayley, viz., postulation for the number $N - r_n$, and postulandum for $q_n$; but on account of the fact that the numbers $N - r_n$ and $q_n$ are not so convenient for dealing with as $r_n$ and $q_n$ it is preferable to have simple terms for the latter. Excess and defect are complementary

numbers, of equal importance, having an intimate reciprocal relation which is most clearly seen in the light of the Riemann-Roch theorem.

We shall use another formula below, which requires some explanation, for proof of which I refer to my paper. Let a $C_l$ and $C_m$ be drawn through the point group $N$, intersecting again in a finite point group $N' = lm - N$, $l$ and $m$ being great enough for this to be possible. Then, if $q_n'$ denote the $n'$-ic defect of $N'$, we shall have

$$q_n' = r_n - 1, \qquad (2)$$

where $$n + n' = l + m - 3,$$

provided $n$ is not less than $l-2$ or $m-2$. Formula (2) implies that $N'$ lies on an $n'$-ic if $r_n \geq 1$, since then $q_n' \geq 0$; and that $N'$ does not lie on an $n'$-ic if $r_n = 0$. These properties are both true.

This formula, which is given by Miss Scott, easily supplies the answer to her question on p. 270: "Having found in any given case that the $N$ points which form the partial intersection of $C_l$ and $C_m$ have an $n$-ic excess $r_n$, is there any way of deciding whether a $C_n$ through $N - r_n$ of these necessarily passes through the remainder?" In hazarding an answer she speaks of *the* $n'$-ic through the $N' = lm - N$ points, apparently not noticing that if $r_n > 1$ formula (2) shows that there is a system of $n'$-ics through $N'$ with freedom $r_n - 1$. "An $n$-ic through $N - r_n$ of the $N$ points passes necessarily through the remainder $r_n$, if the $N - r_n$ points supply $N - r_n$ independent conditions for $n$-ics" (Art. 2, $h$ of my paper); but it is quite possible that this condition should not be fulfilled. In order that the $n$-ic excess of $N$ may be $r_n$ it is necessary and sufficient that the $n'$-ic defect of $N'$ should be $r_n - 1$; and in order that the $n$-ic excess of $N - r_n$ may at the same time be 1, it is necessary and sufficient that the $n'$-ic defect of $N' + r_n$ should be 0, or that the $r_n$ points should lie on an $n'$-ic through the $N'$ points. And as the $n'$-ics through $N'$ have a degree of freedom $r_n - 1$, this only imposes one additional interconnection between the $r_n$ or the $N$ points. For example, let $C_7$ and $C_8$ intersect in 56 points, made up of $N' = 9$ points forming the base of a pencil of cubics, $r = 2$ more points lying on a cubic through the 9, and $N - r = 45$ others. The 9-ic excess of the $N = 47$ points is 2, and the 9-ic excess of the $N - r = 45$ points is 1, so that a 9-ic through the $N - r = 45$ points does not necessarily pass through the remaining 2 of the $N = 47$ points.

Miss Scott justly points out (p. 268) the obscurity of the terms complete, incomplete, and redundant, as applied in my paper to point groups. But these terms, or some equivalent ones, are useful; and a great deal depends on recognizing the distinctions involved in them. If we take away 2 points from the group of 38, mentioned in the second paragraph above, we have left a group of 36 points whose 8-ic excess is 1, while every 8-ic through the 36 points passes through 2 more fixed points. The 38 points form a complete group which cannot, in general, be decomposed into simpler groups; while the 36 points are said to form an incomplete group, because they form part of the complete group 38. But 7-ics through the 36 points pass through the 2 fixed points and 4 others, forming a group of 42 points (the total intersection of $C_6$ and $C_7$). This group of points is also complete; but it is not only complete, it is composite, since it can be decomposed into a group of 4 points on a straight line and the group 38. So also the 47 points mentioned above, which are made up of 45 and 2, form a composite group. A redundant point group, consisting of a non-composite group together with an additional number of general points, is the simplest kind of a composite point group. In order to reduce a point group, by passing two curves through it to intersect again in a less complex group, it is, in general, essential to recognize, and separate out, its constituent groups, if it happens to be composite.

The method of reduction in my paper includes the reduction of redundant but not of other composite point groups. Some notes are given explaining how the appearance of composite point groups may be avoided in the course of reduction; but these are not so much "limitations" as extensions of the method. This is exemplified in the reduction of the point group given below. The object is to find the simplest reduction, and consequent construction, for a point group of assigned characterization, *i. e.*, a point group of which the number of points and the excesses for curves of all orders are given. It seems probable that the simplest point group with an assigned characterization is a non-composite one if such exists, and that, in any case, it is the least composite. I may add here that in my paper I give formulæ for easily calculating the number of independent interconnections of the points of a group if its construction is known. This number is an important one, and serves as an index of the complexity of the group.

On p. 272 Miss Scott gives an example of what she considers to be an impossible point group, viz., $N = 369$ with

excesses 48, 28, 16, 7, 2 for curves of order 24, 25, 26, 27, 28. I proceed to show that a point group with this characterization can be constructed. We begin by supposing that the point group is redundant, *i. e.*, that it consists of a point group $N_0 = 367$, and 2 general points in the plane. We also suppose $N_0$ to lie on a $C_{23}$, and that its 23-ic defect is 1, so that its 23-ic excess is 69. This would still leave it impossible for a $C_{23}$ to pass through $N$. The excesses of $N_0$ for curves of order 24 to 28 will clearly be the same as those of $N$. For the reduction we modify formula (2) by substituting for $q_n'$ its value in terms of $r_n'$ from (1); we then have

$$r_n' = N' + r_n - \tfrac{1}{2}(n'+1)(n'+2), \qquad (3)$$

where $n + n' = l + m - 3$, $N + N' = lm$, $r_n > 0$, and $n$ is not less than $l-2$ or $m-2$. The reduction may be exhibited as follows, the explanation being given after:

$N_0 = 367, r_{28} = 2, r_{27} = 7, r_{26} = 16, r_{25} = 28, r_{24} = 48, r_{23} = 69;$

$N' = 162, r_{15}' = 28, r_{16}' = 16, r_{17}' = 7, r_{18}' = 0, r_{19}' = 0, r_{20}' = 0;$

$N'' = 63, \quad r_{12}'' = 0, \quad r_{11}'' = 1, \quad r_{10}'' = 4;$

$N_0'' = 65, \quad r_{12}'' = 1, \quad r_{11}'' = 3, \quad r_{10}'' = 6;$

$N''' = 35, \quad r_5''' = 15, \quad r_6''' = 10, \quad r_7''' = 5.$

The point group $N'$ is derived from $N_0$ by passing two curves $C_{23}$ through $N_0$ ($q_{23} = 1$). Hence $N' = 23^2 - 367 = 162$; and in applying (3) to find the excesses $r_n'$ of $N'$, we have $l = m = 23$, $n + n' = 43$, and the values corresponding to $n = 28, 27, \cdots, 23$ in the first line are $n' = 15, 16, \cdots, 20$, in the second line. Similarly $N'' = 63$ is derived by passing two curves $C_{15}$ through $N'$ ($q_{15} = 1$), and (3) is again applied for finding the excesses of $N''$. Since $r_{16}' = 0$, $N''$ does not lie on a $C_9$. We have supposed $N''$ to be incomplete, forming part of a complete $N_0'' = 65$. The 10-ic and 11-ic excesses of $N_0''$ will exceed those of $N''$ by 2, and we have supposed the 12-ic excess to be 1. Finally $N''' = 35$ is derived by passing two curves $C_{10}$ through $N_0''$ ($q_{10}'' = 6$). The last point-group $N'''$ is a recognizable one, although, as Miss Scott says, it must be examined with care. $N'''$ consists of 35 general points on a $C_5$, for the excesses of 35 such points for curves of order 5, 6, 7, 8 are 15, 10, 5, 0.

All the steps are reversible. Since two curves $C_{10}$ through $N_0''$ determine $N'''$, so two curves $C_{10}$ through $N'''$ deter-

mine $N_0''$; $N_0''$ deprived of two of its points gives $N''$; two curves $C_{15}$ through $N''$ determine $N'$; and two curves $C_{23}$ through $N'$ determine $N_0$, to which is added any two general points in the plane, giving $N$.

The number of independent interconnections of the $N = 369$, or the $N_0 = 367$, points, when constructed in this way, is 217. The least possible number of interconnections for a point group with the assigned characterization is 215.

If two curves $C_{24}$ are passed through the $N = 369$ points, constructed as above, they determine a group of 207 points which is composite, being made up of 45 points on a conic and the $N' = 162$ points found above. In the actual reduction we have, so to speak, eliminated the 45 points.

If I omitted all reference to the criticism in Miss Scott's last paragraph but one (p. 273) I might be taken as acquiescing in it. The whole question resolves itself into this. Having given a $C_n$ with any number and kind of multiple points can we always find a curve $C_{n'}$ ($n'$ being greater than $n$ if necessary) whose coefficients differ from those of $C_n$ only by infinitely small amounts, and such that at each and every multiple point $A$, of order $p$ on $C_n$ the curve $C_{n'}$ passes through $\frac{1}{2}p(p+1)$ points arbitrarily but generally chosen about and infinitely near to $A$? This is not so much a doubtful matter of opinion as a matter of fact which can be proved or disproved analytically. The convention, which I adopt, of replacing $C_n$ by $C_{n'}$ is an extremely convenient one for the purpose of reasoning geometrically about the intersections of curves, since, for one thing, it enables us to consider the intersection of two curves at a common multiple point as being made up of separate instead of coincident points, just as we consider a tangent to a curve as meeting it in two points at the point of contact. It does not claim to have any other merit or application.

I may add that, since writing the above, I have succeeded in proving that a point group is a possible one if the second differences of its excesses for descending orders of curves are all positive integers, among which zeros may be included; otherwise the point group is impossible.

LONDON,
*June* 6, 1898.

## ELLIOTT'S ALGEBRA OF QUANTICS.

*An Introduction to the Algebra of Quantics.* By EDWIN BAILEY ELLIOTT, M.A., F.R.S. Oxford, Clarendon Press, 1895. 8vo, xiii + 423 pp.

No single book is likely soon to supersede Salmon's Higher Algebra, but it will be replaced gradually by a combination of others, each of narrower scope. Professor Elliott's book is likely to be one of these, and to become equally well known. So far as binary forms are concerned, it worthily signalizes the advance made by English investigators during the past quarter century. To the memoirs of French and German writers before Hilbert there has been ready access through the standard works of Clebsch, Gordan, and Deruyts. There was needed a manual which should introduce the student to the researches of Cayley and Sylvester since 1870, and of Franklin, Hammond, Forsyth, MacMahon, and others of the English school.

A cursory inspection of the book shows that its style is that of the lecture, and therefore easy for the reader. Topical headings are supplied to nearly all sections. Frequent openings are made for geometrical application or illustration; and every student will thank the author for the occasional explicit warnings against the most probable errors and misconceptions. As to the matter, of 415 pages all but 45 are given to binary quantics, although incidental references to forms in more than two variables occur throughout the introductory chapters. Eleven chapters (259 pages) treat questions pertaining to binary forms of all orders; four chapters (110 pages) are given to particular forms and systems, and a final chapter (45 pages) to ternary forms and particularly the cubic. Well-chosen examples, more than 350 in number, distributed through the book double its value.

The first four chapters are entitled: Principles and Direct Methods, Essential Qualities of Invariants, Essential Qualities of Covariants, Cogredient and Contragredient Quantities. The first defines invariants, and gives abundant elementary examples, eliminants, Jacobians, Hessians, discriminants, catalecticants, canonizants. In the second and third, homogeneity and isobarism are shown to be necessary properties. Concomitants of one quantic are made to yield joint concomitants of two or more by substituting for the quantic $f$ a linear aggregate of the same order: $f + \lambda\varphi + \psi\mu$

+ etc., and developing the corresponding concomitant in powers of the parameters $\lambda$, $\mu$, etc. The third closes with the theorem, so nearly self-evident yet always perplexing to the beginner, that a covariant of covariants is a covariant of the fundamental quantics. The fourth chapter exhibits the relation of certain concomitants to others as emanants (polars), whereby invariants of a quantic of any order give rise to covariants of quantics of all higher orders. Next the invariance of the differential operator

$$\left(\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial y_2}-\frac{\partial}{\partial x_2}\cdot\frac{\partial}{\partial y_1}\right)^r$$

is proven, introducing Cayley's hyper-determinant notation as a means of discovering invariants and covariants in unlimited number; and mention is made of Aronhold's symbolic notation. Here we notice a curious slip of the pen (p. 79): "It is symbolic *ab initio*, denoting a binary quantic $(a_0, a_1, a_2, \cdots, a_p)(x, y)^p$ by $(ax+a'y)^p$, where $a^r a'^{p-r}$ means $a_r$, and $a^r a'^s$ has no meaning unless $r+s$ is a multiple of $p$." It should read, of course: "Unless $r+s=p$." Another (p. 78) is more humorous, where the reader is referred to Cayley's memoirs for the "reduction of irreducible systems."

It is not too much to say that these first four chapters are admirable both in plan and in execution, and especially so if regarded from a pedagogic point of view. By reason of their very excellence we may be pardoned if we venture two critical strictures. (*a*) The first is this: The invariant property is in this theory (not so in the Galois theory of equations) a result purely of the form as distinguished from the content. This view can hardly be emphasized too strongly or too often. It is only partially and imperfectly brought out in this book. When by the formation of emanants it is seen that several sets of cogredient variables can figure in a covariant as well as a single set, the student can be led to perceive that the cogredience, hence the linear character of the transformation, is the fundamental premise. When, again, by a process in every way similar (though the similarity is not here exploited), "intermediate" invariants and covariants are derived, the same thesis should be expounded a second time. When finally the hyper-determinant calculus is presented, it need be but a short step to replace each fundamental quantic of $p$th order, $f_p$, by its equivalent according

to Euler's theorem for homogeneous functions:

$$f_p = \frac{1}{p!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^p \cdot f.$$

The concomitants of the quantic appear then as concomitants solely of the *linear operators in the symbolic quantic:*

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^p;$$

or, by application of the emanation process, as concomitants solely of the *linear symbolic quantics* in the operator product:

$$\left(x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y}\right)\left(x_2\frac{\partial}{\partial x} + y_2\frac{\partial}{\partial y}\right)\cdots\left(x_p\frac{\partial}{\partial x} + y_p\frac{\partial}{\partial y}\right)$$

At this point the student would of necessity comprehend the purely formal nature of invariants. At the same time he would see the connection, or better, the identity, between Cayley's hyper-determinant notation and the Aronhold symbolic; and this is certainly the point at which the latter could be explained with least effort.

(*b*) Two of the three processes just specified are well developed by Professor Elliott, but lose most of their theoretic value for lack of the third and for want of emphatic enunciation of the main thesis. Were this third process brought in, and the problem of the inner structure of invariants reduced to that of invariants of a system of *linear* quantics, then it would have been easy to strengthen the framework in a second point of prime importance. The chief merit of the hyper-determinant calculus is not its formal character, but the fact that its method is exhaustive. Not only does it produce a limitless number of covariants, but it produces all possible covariants. This converse theorem, that every rational covariant can be produced by an aggregate of hyper-determinant operators, ought evidently to be the first climactic theorem in an English treatise. The fact that Cayley did not in his early memoirs make this a cardinal point ought not to obscure its significance in the light of later knowledge.* Although we are told (p. 78) that this

* See Meyer: "Bericht über den gegenwärtigen Stand der Invariantentheorie," *Jahresberichte der deutschen Math. Vereinigung*, vol. 1, p. 98, first half-page.

method "did not, in its originator's form, succeed in establishing the finiteness of complete systems of irreducible covariants in general; that triumph was reserved for a later method, etc.," yet the discovery of it was none the less a noteworthy achievement, and its unrealized potency may most properly be acknowledged and elucidated.

This opportunity for a principal result being apparently neglected, an equivalent is given in the fifth chapter under the caption: "Invariant functions of the differences of roots" (p. 93). It is assumed as known from the elementary theory of equations, that any function of the roots which is not altered when all roots are increased or decreased equally is a function only of the differences of the roots. Every rational invariant is such a function multiplied by some power of the leading coefficient of the equation, and is symmetric in the roots. This lemma gives a system of Diophantine equations satisfied by the exponents of the differences in any term of the invariant function of roots. Hereupon Hermite's law of reciprocity could be employed to prove the sufficiency of hyper-determinants to represent all covariants, as the author points out somewhat later (p. 161).

From this point on, in the general theory, two principal results are to be reached. The first is Cayley's theorem, proved by Sylvester, concerning the exact number of independent covariants of given degree in the coefficients and of given order in the variables. Denoting weight, degree, and order of the fundamental quantic by $w$, $i$, $p$ respectively, the exact number is denoted in partition-symbols by

$$(w\,;\,i,\,p) - (w-1\,;\,i,\,p).*$$

The second is Gordan's theorem, that all covariants of any given set of binary quantics are rational integral functions of a finite number of such, constituting a "complete form-system."

The proof of the first of these theorems requires the discussion of the two differential equations that covariants must satisfy, and of the corresponding differential operators (annihilators of covariants) $\Omega$ and $O$, with their alternant: $\eta = \Omega O - O\Omega$. The name semivariants is given to all homogeneous isobaric functions of the coefficients of the quantic which are annihilated by $\Omega$; and those are found to be the coefficients of leading terms of covariants, which in turn are uniquely determined from the semi-invariant lead-

* See Salmon's Higher Algebra, 4th ed., p. 132, where $p = n$ and $i = \theta$.

ers. Chapters VI and VII are devoted to this relation, and to the exact enumeration of semiinvariants of given degree and excess, whence follows the proof of the number of independent covariants.

In closing Chapter VII Hermite's law of reciprocity is neatly demonstrated. Chapter VIII is an introduction to the study of generating functions both for the number of covariants of given type, and for the number and types of those forming a complete system for a given quantic. The example most fully treated, that of the system of concomitants of a cubic, is so attractively presented as to make this difficult and neglected subject appear one of the most interesting. The same may be said of the Chapters X and XI, on Protomorphs and Perpetuants.

For the second principal theorem, the so-called Gordan theorem, the proof selected is that extremely simple one given by Hilbert in vol. 30 of *Math. Annalen.* As illustrations the invariants of the cubic and quartic are analyzed, where Hilbert's limit to the number of independent invariants is seen to be largely in excess of the true number. Although this mode of proof has no extension to ternary and higher quantics, its simplicity and elegance fully justify its adoption in preference either to Gordan's proof, or to Hilbert's own more general method.

Of the four chapters on particular quantics, one deals with canonical forms, canonizants, and catalecticants. Elliott's discussion is probably more elementary, but Salmon's chapter on the same topic should be read in connection with it. It is questionable whether the canonizant, for example, might not be introduced more appropriately by its invariant differential equation, reserving till later its relation to a canonical form. This would seem particularly desirable as a natural introduction of irrational covariants. The quintic and sextic have a chapter, simultaneous quantics another, and the invariants called by this author *Boolians* are accorded the closing place among binary quantics. A brief but valuable chapter on ternary differential operators and the invariants of a ternary cubic is added, raising the hope that the author may purpose a similar volume on ternary quantics, or at least the cubic and quartic, as a sequel to this. The appearance of any work of so high merits as this Algebra of quantics is certain to result in a renewed and more widespread activity in the domain traversed.

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## HADAMARD'S GEOMETRY.

*Leçons de Géométrie élémentaire.* Par JACQUES HADAMARD. Paris, Colin et Cie. 8vo, xvi+308 pp.

THE writing of good books for school use is, one can readily imagine, not an easy matter in so old a science as mathematics. On the one hand is a vigorous tradition, on the other the desire to effectively prepare the way for a real knowledge of the modern science. It is, then, good for us to be able to obtain an authoritative indication of the French point of view, from the series of elementary text-books now being published under the direction of M. Darboux.

M. Hadamard contributes a book on plane geometry which is intended for such beginners as expect to pursue serious studies in mathematics. While retaining the accepted view that the training in reasoning is especially to be kept in mind, he contrives, by selection and arrangement of matter, and by style, to let appear the intrinsic fascination of geometry. The rules of the game do not spoil the game itself; and these rules are explained, not dogmatically stated.

The work begins with a short introduction. Then follow books on the straight line, on the circle, on similitude, and on areas. The third book is followed by an important section headed Compléments du Livre III, which contains the bulk of the post-Euclidian part such as the theories of anharmonic ratios, pole and polar, and inversion. It would seem more convenient to call this section Book IV. Other modern ideas, for instance the displacement of a figure or the radical axis, are worked into the books themselves.

The whole, so far, is divided into chapters of proper length. Then come four valuable notes on methods in geometry, on Euclid's postulate, on the circles touching three given circles, and on the notion of area. Of exercises and problems there are 422.

What appears of the body of Euclid's doctrine is freely scattered; it reminds us (if we may say so) of a royal mummy used for fertilization.

One feature which is especially admirable is that when the author passes from the strict geometrical reasoning on the figures themselves to arithmetical reasoning he emphasizes the transition; thus the beginner realizes the difference.

We may make one small objection. The circle is said to be a *region*; the circumference being the boundary line, as in Euclid. This usage is not retained in later life and might well be given up. It is not used consistently throughout the book, the circle being spoken of where the curve is intended.

Enough has been said, we hope, to show, without insisting on details, the value of the work to the teacher. Even the rigid Euclidian would find side-lights on the meaning of the old classic; perhaps more than in many school editions.

FRANK MORLEY.

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## FURTHER NOTE ON EULER'S USE OF $i$ TO REPRESENT AN IMAGINARY.

IN a communication recently received from M. Enestrom he gave it as his impression that attention was called to Euler's use of $i$ for $\sqrt{-1}$ some years ago in the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht.*

On investigation I find that one of the principal passages quoted by me in the March number of the BULLETIN was referred to by Heymann and Ackermann in this *Zeitschrift*, vol. 17 (1886), pp. 509, 580.

A careful examination of such of Euler's memoirs as are found in the library of the United States Naval Observatory, undertaken at my request by Mr. George K. Lawton, has revealed no other papers, either before or after 1777, in which Euler uses $i$ for $\sqrt{-1}$.

Further information would be gladly welcomed.

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# NOTE ON NAPIER'S RULES OF CIRCULAR PARTS.

BY PROFESSOR EDGAR ODELL LOVETT.

THE object of this note is to call attention to Napier's conception of Napier's rules of circular parts in the theory of the right spherical triangle. Text-books and treatises are considerably at variance as to the importance and status of these rules. They are continually referred to as mere mnemonics requiring independent proof in each case; and not infrequently their rôle in this capacity is discouraged, analogies with corresponding formulæ in the plane being offered as preferable substitutes. What follows shows that these much abused individuals are entitled to more generous consideration and should be invested with the dignity of a theorem. The circumstance of their various misinterpretations emphasizes again that the student of mathematics should trust no middle man, but go with his own head to original sources, to the masters themselves. Second-hand ideas are as full of bacteria as are second-hand books and clothes.

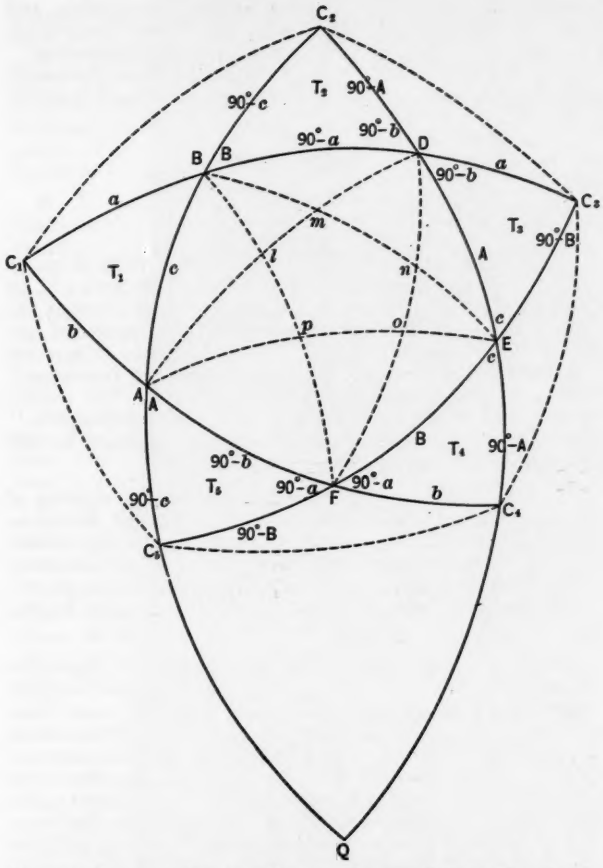
Let $ABC_1$ be a right spherical triangle whose right angle is at $C_1$; draw the great circles which coincide with the sides of the triangle and those whose poles are the vertices $A$ and $B$. These five great circles form a spherical pentagon surmounted by five right spherical triangles. The uniformity of Napier's rules lies in the fact that all five of these associated triangles, though possessing different natural parts, have one and the same set of circular parts.

A little attention to the geometry of this beautiful figure yields the following geometrical interpretations of the rules of Napier, the first two of which were given by Napier in the original presentation of the rules,* and the first of which was rediscovered by Ellis.†

Napier's rules express 1° properties common to the members of the system of five right-angles $T_1$, $T_2$, $T_3$, $T_4$, $T_5$; 2° characteristics which the individuals of the family of five quadrantal triangles $ABD$, ···, have in common; 3° relations among the parts of the self-polar pentagon $ABDEF$; 4° relations in the elements of the pentagon $C_1C_2C_3C_4C_5$; 5° relations among the sides and angles of the pentagon

*See Napier—Logarithmorum canonis descriptio, Lugduni, apud Barth. Vincentium, MDCXX, Lib. II., Cap. IV., pp. 31 et seq.

†See The mathematical and other writings of Robert Leslie Ellis, edited by William Walton, Cambridge, 1863, pp. 328 et seq.




*lmnop*; 6° properties peculiar to the quadrilaterals of the system of five trirectangular quadrilaterals $FC_2QC_4$, ···; 7° relations common to the members of the family of five complete quadrilaterals $ABC_2DEm$, ···, the latter family including as particular cases the system *ABDE*, ···; *ADEB*, ···; ···.

It will be further remarked that any one figure of the en-

semble determines all the other members completely, and that the degenerate forms of the aggregate when the original triangle is birectangular or trirectangular are interesting.

Mauduit's rules and those of our countryman, Nathaniel Bowditch, are susceptible of similar extensions and interpretations.

PRINCETON, NEW JERSEY,
*April* 12, 1898.

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## NOTES.

THE Colloquium in connection with the Fifth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Harvard University, Cambridge, Mass., during the week beginning Monday, August 22d. The course of lectures offered by Professor Osgood is entitled: "On some methods and problems of the general theory of functions." The title of Professor Webster's course is: "The partial differential equations connected with wave propagation." Details in regard to the Colloquium are contained in the circular issued by the committee in charge.

THE preliminary programme for the coming meeting of Section A of the American Association for the Advancement of Science comprises thirty papers, including several reports on progress in different branches of mathematics. The meeting of the Association takes place August 22–27. Professor E. E. Barnard is the Vice-President, and Professor J. McMahon the Acting Secretary of Section A.

THE Göttingen Royal Society has taken active measures to resume and complete the publication of Gauss' collected works, the first six volumes of which appeared more than twenty years ago. The *Nachrichten* (No. 1, 1898) contains an account, by Professor F. Klein, of the plan in contemplation. The death of Professor Schering, to whom the publication was originally entrusted, has necessitated a distribution of the remaining work among a considerable number of editors. The astronomical material has been assigned to Professor M. Brendel, who also undertakes the general editorial supervision of the whole work. The other editors and their subjects are: Professor R. Fricke, analysis and the theory of numbers; Professor P. Stäckel, geometry; Professors Börsch and Krüger, geodesy; Professor Wiechert, mathematical physics. The work is to be completed in three volumes and a supplementary volume. Vol. 7 will be de-

voted exclusively to astronomy, and will contain all the astronomical material not already included in vol. 6; vol. 8 will contain the scientific supplement to the earlier volumes; vol. 9 will be devoted to the biographical material. The supplementary volume will contain the general index. It is hoped that the entire work may de completed in three years. The sale of the first six volumes has been transferred to B. G. Teubner.

M. L. LAUGEL s French translation of B. Riemann's Mathematical Works has just appeared from the press of Gauthier-Villars of Paris. PROFESSOR M. CANTOR'S great work on the history of mathematics (up to the year 1758) has now reached its completion with the issue of the third part of the third volume. The first part of Professor T. Reye's Geometry of Position, translated by PROFESSOR T. F. HOLGATE of Northwestern University has just been published by The Macmillan Company.

PROFESSOR G. H. DARWIN'S lectures at the Lowell Institute, Boston, 1897, will soon appear in book form under the title: "The tides and kindred phenomena in the solar system." John Murray is the publisher.

DR. G. W. HILL is preparing a work on Planetary Motion, which is to be published by G. P. Putnam's Sons as a volume of their Science Series.

A NUMBER of printed papers and particularly two works of the mathematician Antonio Monforte, discovered by Professor Frederico Amodeo, furnish abundant proof that the Royal Academy of Sciences of Naples, hitherto supposed to date from about 1732, was in active existence as early as 1698.

PROFESSOR SIMON NEWCOMB has resumed his professorship of mathematics at the Johns Hopkins University. ASSOCIATE PROFESSOR A. S. CHESSIN has resigned his position.

DR. G. W. HILL has been appointed lecturer in celestial mechanics in Columbia University. A fund of five thousand dollars has been given to maintain this lectureship by Miss Catherine W. Bruce.

PROFESSOR E. B. FROST, of Dartmouth College, has been appointed professor of astrophysics at the Yerkes Observatory.

PROFESSOR ALEXANDER ZIWET, of the University of Michigan, has gone abroad for a year on leave of absence.

MR. E. A. LYMAN, instructor in mathematics at the University of Michigan, has been appointed professor of mathematics at the Michigan State Normal College, to succeed Professor D. E. Smith. MR. WM. H. BUTTS, formerly of the Orchard Lake Military Academy, and MR. A. W. WHITNEY, of the University of Nebraska, have been appointed instructors in mathematics at the University of Michigan.

PROFESSOR E. J. TOWNSEND, of the University of Illinois, will spend two years abroad on leave of absence. PROFESSOR G. W. MYERS, associate professor of mathematics and astronomy, has been promoted to a professorship of astronomy and applied mathematics.

DR. L. W. DOWLING has been promoted to an assistant professorship of mathematics in the University of Wisconsin.

DR. G. A. MILLER, of Cornell University, will conduct a seminar in permutations at the University of Chicago in the coming summer term.

PROFESSOR J. B. SHAW, of Illinois College, has resigned his position.

PROFESSOR SOPHUS LIE, of the University of Leipzig, has accepted a call to the University of Christiania.

MR. A. E. H. LOVE, F.R.S., and MR. R. PENDLEBURY, fellows and lecturers of St. John's College, Cambridge, have been appointed university lecturers in mathematics.

PROFESSORS H. A. LORENTZ, of Leyden, and ÉMILE PICARD, of Paris, have been elected honorary members of the London Mathematical Society to succeed the late Professors Brioschi and Hertz.

AT the recent meeting of the Royal Society PROFESSOR E. W. BROWN, MR. H. F. BAKER, and Mr. M. H. TAYLOR were elected fellows of the Society.

THE University of California has conferred the degree of LL.D. on PROFESSOR J. M. SCHAEBERLE in recognition of his services to the Lick Observatory.

THE death is announced of PROFESSOR SOUILLART, professor of astronomy in the University of Lille and correspondent in the section of astronomy of the Paris Academy of Sciences.

THE REV. PERCIVAL FROST, F.R.S., D.Sc., died June 6th, at Cambridge, England, in his 81st year. In 1839 Dr. Frost was second wrangler and first Smith's prizeman at Cambridge, 1839 to 1841 fellow of St. John's College, 1847 to 1859 mathematical lecturer at Jesus College, 1859–1889 mathematical lecturer at King's College, Cambridge. Dr. Frost was the author of numerous papers appearing in the various British mathematical journals and of the well-known treatises on "Curve tracing," "Solid geometry," and, "The first three sections of Newton's Principia."

CORNELL UNIVERSITY. During the academic year 1898–99 the following advanced courses will be offered by the department of mathematics. each course occupying either two or three hours a week as indicated, and continuing through the year unless otherwise stated:—By Professor WAIT: Advanced analytic geometry, (*a*) plane, three hours, (*b*) solid, two hours; Advanced differential calculus, three hours, first two terms.—By Professor JONES: Higher algebra and trigonometry, three hours; Probabilities, etc., two hours.—By Professor MCMAHON: Higher plane curves, two hours; Quaternions, two hours; Potential, and Spherical harmonics, two hours; Mathematical theory of sound, two hours:—By Professor TANNER: Binary quantics, two hours; Theoretical mechanics, two hours; German readings, two hours.—By Dr. MURRAY: Differential equations, three hours; Finite differences, etc., two hours; Astronomy, two hours.—By Dr. HUTCHINSON: Advanced integral calculus, two hours; Elliptic functions, three hours, first term, Surface and twisted curves, three hours, last two terms:—By Dr. SNYDER: Projective geometry, three hours; General function theory, three hours; Line geometry, two hours:—By Dr. MILLER: Substitution groups, three hours, first term; Continuous groups, three hours, last two terms; Theory of numbers, two hours.

The Oliver Mathematical club will hold fortnightly conferences.

JOHNS HOPKINS UNIVERSITY. During the coming year PROFESSOR SIMON NEWCOMB, LL.D., F.R.S., lately director of the U. S. nautical almanac, will resume his superintendence of the work in mathematics and astronomy. He will give especial attention to students desiring to pursue advanced work in celestial mechanics. Near the beginning of the year, Professor Newcomb hopes to give a short course of lectures on the Encyclopædia of the Mathematical Sci-

ences,—a course which will be open to graduate students in physics and astronomy as well as to those in mathematics.

The following advanced courses are announced by the department of mathematics, each course extending through the year unless otherwise noted :—Professor CRAIG : Theory of surfaces, three hours ; Advanced theory of functions and differential equations, three hours ; Mathematical conference, weekly. — By Dr. COHEN : Elementary theory of functions, two hours ; Theoretical mechanics, two hours ; Substitution groups, two hours, first half-year ; Elliptic functions, two hours, second half-year.

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## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

ANDOYER (H.). Leçons élémentaires sur la théorie des formes et ses applications géométriques. (Autographié.) Paris, Gauthier-Villars, 1898. 4to. 6 and 184 pp. Fr. 8.00

BAER (K.). Die Kugelfunction als Lösung einer Differenzengleichung. Berlin, Mayer and Müller, 1898. 4to. 25 pp. Mk. 1.50

BAKER (W. M.). Examples in analytical conics for beginners. London, Bell, 1898. 8vo. 96 pp. 2s. 6d.

BOURGET (H.). Sur une classe particulière de groupes hyperabéliens. [Diss.] Paris, Gauthier-Villars, 1898. 4to. 95 pp.

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Dritter (Schluss-) Band : Von 1668–1758. Leipzig, Teubner, 1898. 14 pp. and pp. 473 to 893. Mk. 20.00

ESCARY (M.). Remarques concernant les formules fondamentales de la trigonométrie. Paris, Hermann, 1898. 8vo. 11 pp. Fr. 0.60

HOLGATE (T. F.). See REYE (T.).

KLUG (L.). Die Configuration des Pascal'schen Sechseckes im Allgemeinen und in 4 speciellen Fällen. Wien, Eisenstein, 1898. 8vo. 132 pp., 3 plates. Mk. 3.00

LA COUR (P.). Historisk Matematik. 2te gennemsete udgave. Haefte 1. Copenhagen, 1898. 8vo. 32 pp. Mk. 0.80

LAPLACE. Oeuvres complètes de Laplace, publiées sous les auspices de l'Académie des sciences. Vol. XII. Paris, Gauthier-Villars, 1898. 4to. 574 pp. Fr. 20.00.

LEROY (E.). Sur l'intégration des équations de la chaleur. Paris, Gauthier-Villars, 1898. 4to. 264 pp.

MILLER (G. A.). On the primitive substitution groups of degree sixteen. (*American Journal of Mathematics*, Vol. XX., No 3.) 4to. Pp. 229–241.

—— On the perfect groups. (*American Journal of Mathematics*, Vol. XX., No. 3.) 4to. Pp. 277–282.

MOLK (J.). See TANNERY (J.).

RANKIN (T. T.). Complete solutions to papers in mathematics. Second stage, 1887–1897. 4th edition. London, Simpkin, 1898. 8vo. 2s.

RAZZABONI (A.). Delle superficie nelle quali un sistema di geodetiche sono curve del Bertrand. Nota I. Bologna, Gamberini, 1898. 8vo. 9 pp. Nota II. Bologna, Gamberini, 1898. 8vo. 17 pp.

REYE (T.). Lectures on the geometry of position. Translated and edited by T. F. Holgate. Part I. New York, Macmillan, 1898. 8vo. 19 and 248 pp. Cloth. $2.25

SCHIMPF (E.). Zur Definition der Konvergenz der unendlichen Reihen und der unendlichen Produkte. Mehrfache Grenzgleichungen. Grenzgleichungen periodischer Reihen. [Progr.] Berlin, Mayer & Müller, 1898. 4to. 3 and 30 pp. Mk. 1.00

TANNERY (J.) et MOLK (J.). Eléments de la théorie des fonctions elliptiques. Tome III: Calcul intégral. Partie I: Théorèmes généraux; inversions. Paris, Gauthier-Villars, 1898. 8vo. 8 and 267 pp. Fr. 8.50

THOMAE (J.). Elementare Theorie der analytischen Funktionen einer complexen Veränderlichen. 2te Auflage. Halle, Nebert, 1898. 4to. 8 and 150 pp. Mk. 9.00

## II. ELEMENTARY MATHEMATICS.

HOLZINGER (F. S.). Lehrbuch der praktischen Arithmetik für höhere Handelsschulen und zum Selbstunterricht. 2te unveränderte Auflage. Braunschweig, 1898. 8vo. Mk. 3.00

—— Lehrbuch der politischen Arithmetik. 2te unveränderte Auflage. Braunschweig, 1898. 8vo. 9 and 156 pp. Mk. 3.00

LEYSSENNE (P.). Tratado de aritmetica teórica y práctica para uso de las escuelas normales de profesores et profesoras, etc. Traducida de la 13a edicion. Paris, Bouret, 1898.

MURRAY (E.). Higher arithmetic and mensuration for all civil service, intermediate, army, university and teachers' examinations with solutions to the most difficult questions set at these examinations. London, Blackie, 1898. 8vo. 304 pp. 3s. 6d.

NICHOLSON (J. W.). Elements of plane and spherical trigonometry. New York, Macmillan, 1898. 8 and 101 pp. Cloth. $0.90
The same with logarithmic and trigonometric tables. 8 and 101 and 61 pp. $1.10

ROBINSON'S new higher arithmetic for high schools and academies and mercantile colleges. New York, American Book Co., 1898. 8vo. 527 pp. Half leather. $1.00

SCHUBERT (H.). Arithmetik und Algebra. 2te Auflage. Leipzig, 1898. 12mo. 171 pp. Cloth. Mk. 0.80

ULRICH (G.). Ausführliches Lehrbuch der Arithmetik und Algebra, mit zahlreichen Uebungsaufgaben und Auflösungen. Berlin, 1898. 8vo. 7 and 400 pp. Mk. 3.00

## III. APPLIED MATHEMATICS.

BEAZELEY (A.). Tables of tangential angles and multiples; or setting out curves. 6th edition, revised. 50 cards in case. London, Lockwood, 1898. 3s. 6d.

BOTT (P.). Graphische Darstellung elektrischer Wechselströme. Berlin, Gaertner, 1898. 4to. 35 pp. Mk. 1.00

CARON (J.). Cours de géométrie descriptive. Géométrie cotée. Paris, 1898. 8vo. Fr. 6.50

CARONNET (T.). Problèmes de mécanique, à l'usage des classes de mathématiques élémentaires et de première-sciences, et des candidats au baccalauréat et aux écoles du gouvernement; Cinématique et dynamique. 2d part. Paris, Nony, 1898. 8vo. Pp. 185–401.

DARCHEZ (V.). Nouveau cours de dessin géométrique. Paris, 1898. 4to. With 67 plates. Fr. 4.00

FUHRMANN (A.). Anwendungen der Infinitesimalrechnung in den Naturwissenschaften, im Hochbau und in der Technik. Lehrbuch und Aufgabensammlung. 3ter Theil, 1te Hälfte. Berlin, Ernst, 1898. 8vo. pp. 1–180. Mk. 5.50

JUDE (R. H.). First stage magnetism and electricity (treated from the standpoint of potential and potential-gradient). For elementary examination of the Science and art department. With numerous exercises and examination questions. London, Clive, 1898. 8vo. 358 pp. 2s.

KENNEDY (A. B. W.). The mechanics of machinery. 3d edition, revised and corrected. London, Macmillan, 1898. 8vo. 668 pp. 8s. 6d.

LAURENSTEIN (R.). Leitfaden der Mechanik. 3te Auflage. Stuttgart, 1898. 8vo. Cloth. Mk. 4.00

LÉVY (M.). Leçons sur la théorie des marées. Partie I: Théories élémentaires. Formules pratiques de la prévision des marées. Paris, Gauthier-Villars, 1898. 4to. Fr. 14.00

LORENZ (F.). Oeuvres scientifiques. Revues et annotées par H. Valentiner. Publiées aux frais de la Fondation Carlsberg. Tome I, Fasicule 2. Copenhagen, 1898. 8vo. pp. 211 to 624. Mk. 6.50

NERNST (W.) und SCHOENFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung, mit besonderer Berücksichtigung der Chemie. 2te vermehrte und verbesserte Auflage. München, Wolff, 1898. 8vo. 12 and 339 pp. Mk. 9.00

SCHAFFERS (V.). Essai sur la théorie des machines électriques à influence. Paris, Gauthier-Villars, 1898. Fr. 3.00

SCHOENFLIES (A.). See NERNST (W.).

VALENTINER (H.). See LORENZ (F.).

WILSON (C. A. C.). Electro-dynamics. The direct current motor. London, Longmans, 1898. 8vo. 306 pp. 7s. 6d.

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## ERRATA.

p. 332, line 5, *for* 1897 *read* 1898
p. 382, " 38–9, " Mathematical Conference at Chicago " Chicago Section
p. 387, " 12, " $C^3$ " $C_6$
p. 388, " 5, " group is " proof is
" " 1, *add:* $h_1$ or else.

# SEVENTH ANNUAL LIST OF PAPERS

## READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

BLAKE, E. M. Note upon a Representation in Space of Ellipses drawn by an Ellipsograph. Read (Chicago) April 24, 1897. *Annals of Mathematics*, Vol. 11, No. 6, pp. 195–196; Oct., 1897.

BÔCHER, M. The Roots of Polynomials which Satisfy Certain Linear Differential Equations of the Second Order. Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 6, pp. 256–258; March, 1898.

—— The Theorems of Oscillation of Sturm and Klein (First Paper). Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 7, pp. 295–313; April, 1898.

—— The Theorems of Oscillation of Sturm and Klein (Second Paper). Read Feb. 26, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 8, pp. 365–376; May, 1898.

—— Note on Poisson's Integral. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 9, pp. 424–426; June, 1898.

BOLZA, O. Concerning the Cubic Involution and the Cubic Transformation of Elliptic Functions. Read Aug. 16, 1897 and (Chicago) Dec. 30, 1897. *Mathematische Annalen*, vol. 50, No. 1, pp. 68–102; Dec., 1897.

BOUTON, C. L. Some Examples of Differential Invariants. Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 7, pp. 313–322; April, 1898.

BROWN, E. W. Note on the Steering of an Eight-oared Boat. Read Oct. 30, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 2, pp. 77–78; Nov., 1897.

CHESSIN, A. S. Note on Hyperelliptic Integrals. Read Oct. 30, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 3, pp. 93–96; Dec., 1897.

DICKSON, L. E. Orthogonal Group in a Galois Field. Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 5, pp. 196–200; Feb., 1898.

—— Systems of Simple Groups Derived from the Orthogonal Group. Read (Chicago) Dec. 30, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 8, pp. 382–389; May, 1898. *Proceedings of the California Academy of Sciences*, vol. 1, No. 4, pp. 29–46; March, 1898.

—— The St ucture of the Hypoabelian Groups. Read (Chicago) April 9, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 495–510; July, 1898.

ENGBERG, C. C. On the Triple Focus of a Cartesian. Read (Chicago) Dec. 31, 1898. *Annals of Mathematics*, vol. 12, No. 2, pp. 54–56; May, 1898.

HATHAWAY, A. S. Quaternions as Numbers of Four-Dimensional Space. Read (Chicago) April 24, 1897. *Bulletin of the American Mathematical Society*, vol 4, No. 2, pp. 54–57 ; Nov., 1897.

HAWKES, H. E. Limitations of Greek Arithmetic. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 530–535 ; July, 1898.

HILL, G. W. On Intermediary Orbits in the Lunar Theory. Read Oct. 30, 1897. *Astronomical Journal*, vol. 18, No. 11, pp. 81–87 ; Nov., 1897.

HILL, J. E. On Three Septic Surfaces. Read Aug. 31, 1896. *American Journal of Mathematics*, vol. 19, No. 4, pp. 289–311 ; Oct., 1897.

HOLGATE, T. F. A Geometrical Locus Connected with a System of Coaxial Circles. Read Aug. 16, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 2, pp. 63–67 ; Nov., 1897.

HUTCHINSON, J. I. On the Reduction of Hyperelliptic Functions ($p=2$) to Elliptic Functions by a Transformation of the Second Degree. Read Aug. 16, 1897. Doctor Dissertation, University of Chicago, 1897, pp. 40.

—— Note on the Tetrahedroid. Read Feb. 26, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 7, pp. 327–329 ; April, 1898.

LOVETT, E. O. Note on the Invariants of $n$ Points. Read April 24, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 2, pp. 58–59 ; Nov., 1897.

—— Certain Classes of Point Transformations in the Plane. Read May 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 3, pp. 97–107 ; Dec., 1897.

—— Note on the Fundamental Theorems of Lie's Theory of Continuous Groups. Read Oct. 30, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 2, pp. 59–63 ; Nov., 1897.

—— Certain Invariants of a Quadrangle by Projective Transformation. Read Dec. 29, 1897. *Annals of Mathematics*, vol. 12, No. 3, pp. 79–86 ; June, 1898.

—— Point Transformations in Elliptic Coördinates of Circles having Double Contact with a Conic. Read Dec. 29, 1897. *American Journal of Mathematics*, vol. 20, No. 3, pp. 242–244 ; July, 1898.

—— Note on the Infinitesmal Projective Transformation. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 515–519 ; July, 1898.

—— Infinitesimal Transformations of Concentric Conics. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 520–524 ; July, 1898.

MCCLINTOCK, E. Further Researches in the Theory of Quintic Equations. Read Aug. 17, 1897. *American Journal of Mathematics*, vol. 20, No. 2, pp. 157–192 ; April, 1898.

MERRIMAN, M. The Probability of Hit when the Probable Error in Aim is Known ; With a Comparison of the Probabilities of Hit by the Methods of Independent and Parallel Fire from Mortar Batteries. Read Oct. 30, 1897. *Journal of the U. S. Artillery*, vol. 8, No. 2, pp. 129–141 ; Sept.–Oct., 1897.

MILLER, G. A. On the Commutator Groups. Read Aug. 17, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 4, pp. 135–139; Jan., 1898.

—— On the Limit of Transitivity of the Multiply Transitive Substitution Groups That do not Contain the Alternating Group. Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 4, pp. 140–143; Jan., 1898.

—— On the Perfect Groups. Read (Chicago) Dec. 30, 1897. *American Journal of Mathematics*, vol. 20, No. 3, pp. 277–282; July, 1898

—— On an Extension of Sylow's Theorem. Read Feb. 26, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 7, pp. 323–327; April, 1898.

—— On the Hamilton Groups. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 510–515; July, 1898.

—— On the Supposed Five-fold Transitive Function of 24 Elements and $19! \div 48$ Values. Read April 30, 1898. *Messenger of Mathematics*, vol. 27, No. 12, pp. 187–190; April, 1898.

MOORE, E. H. Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holohedrically Isomorphic with the Symmetric and the Alternating Substitution Groups on $k$ Letters. Read (Chicago) Jan. 1, 1897. *Proceedings of the London Mathematical Society*, vol. 28, No. 597, pp. 357–366; Dec., 1896.

—— Concerning Regular Triple Systems. Read Aug. 17, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 1, pp. 11–16; Oct., 1897.

MORLEY, F. A Construction by the Ruler of a Point Covariant With Five Given Points. Read Dec. 30, 1896. *Mathematische Annalen*, vol. 49, Nos. 3–4, pp. 596–600.

—— A Generating Function for the Number of Permutations with an Assigned Number of Sequences. Read May 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 1, pp. 23–28; Oct., 1897.

NEWCOMB, S. The Philosophy of Hyperspace. [Presidential Address.] Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 5, pp. 187–195; Feb., 1898. *Science*, vol. 7, No. 158, pp. 1–7; Jan. 7, 1898.

NEWSON, H. B. Continuous Groups of Circular Transformations. Read April 24, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 3, pp. 107–121; Dec., 1897.

—— On Hessians, and Steinerians of Higher Orders in Geometry of One Dimension. Read (Chicago) April 24, 1897. *Annals of Mathematics*, vol. 11, Nos. 3–4, pp. 121–128; Feb.–April, 1897.

—— Normal Forms of Projective Transformations. Read (Chicago) Dec. 30, 1897. *Kansas University Quarterly*, vol. 7, No. 3, pp. 125–141; July, 1898.

OSGOOD, W. F. Example of a Single-Valued Function with a Natural Boundary, whose Inverse is also Single-Valued. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 9, pp. 417–424; June, 1898.

PIERPONT (J.). Early History of Galois' Theory of Equations. Read Feb. 26, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 7, pp. 332–340; April, 1898.

PORTER, M. B. Note on the Roots of Bessel's Functions. Read Feb. 26, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 6, pp. 274–275; March, 1898.

ROE, E. D. Note on the Integral and Integro-Geometric Series. Read Sept. 1, 1896. *Annals of Mathematics*, vol. 11, No. 6, pp. 184–194; Oct., 1897.

SAUREL, P. L. Note on Integrating Factors. Read Feb. 26, 1898 *Bulletin of the American Mathematical Society*, vol. 4, No. 7, pp. 329–331; April, 1898.

SCOTT, C. A. Studies in the Transformation of Plane Curves. Read April 25, 1896. *Quarterly Journal of Mathematics*, vol. 29, No. 116, pp. 329–381; March, 1898.

—— Note on Linear Systems of Curves. Read Oct. 30, 1897. *Nieuw Archief voor Wiskunde*, 2d Series, vol. 3.

SNYDER, V. Geometry of Some Differential Expressions in Hexaspherical Coördinates. Read Aug. 16, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 4, pp. 144–154; Jan., 1898.

STARKWEATHER, G. P. A Solution of the Biquadratic by Binomial Resolventa. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 524–528; July, 1898.

STRONG, W. M. Is Continuity of Space Necessary to Euclid's Geometry? Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 9, pp. 443–448; June, 1898.

VAN VLECK, E. B. On the Polynomials of Stieltjes. Read April 30, 1898. *Bulletin of the American Mathematical Society*, vol. 4, No. 9, pp. 426–438; June, 1898.

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